Network Routing under Strategic Link Disruptions

Mathieu Dahan Saurabh Amin

Massachusetts Institute of Technology

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FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

Outline

Network routing in the face of disruptions

1. Network flow routing
   - Max-flow problem [Fulkerson '56]
   - Max-flow min-cut theorem [Fulkerson '56]
   - Max-flow with minimum transportation cost [Edmonds and Karp '72]
2. Network vulnerability
   - Sequential games: network interdiction [Washburn '95]
   - Simultaneous games [Wooders '10], [Gueye '12]
   - Vulnerability indices [Gueye '12]

(Q): Network routing when the operator faces strategic link disruptions ?

Max-flow problem

$$(\mathcal{P}_1): \quad \text{maximize} \quad F(x) \quad \text{subject to} \quad x \in \mathcal{F},$$

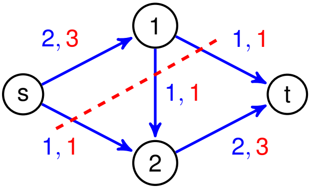
- $F(x)$: Value of flow x

Max-flow with min-transportation cost

$$(\mathcal{P}_2): \quad \text{minimize} \quad C_1(x) \quad \text{subject to} \quad x \in \mathcal{F}, \quad F(x) \geq F(x'), \quad \forall x' \in \mathcal{F},$$

- $C_1(x)$: Cost of transporting flow x

Max-flow min-cut theorem: the maximum value of an $s-t$ flow is equal to the minimum capacity over all $s-t$ cuts.



Our focus

(Q): Network routing when the operator faces strategic link disruptions ?

- We formulate a simultaneous non-zero sum game
  - Both transportation and attack costs
  - Attacker simultaneously disrupts multiple edges
  - Defender strategically chooses a flow but no re-routing after attack.
- Main contributions
  - Structural insights on the set of Nash equilibria
  - Relation to classical network routing problems
  - Network vulnerability under strategic attacks

Game

$\Gamma := \langle \{1,2\}, (\mathcal{F}, \mathcal{A}), (u_1, u_2) \rangle$

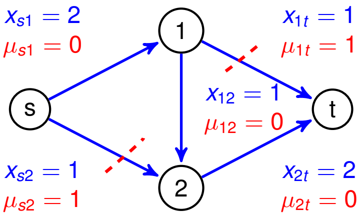
- Directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and for every $(i,j) \in \mathcal{E}$:
  - Edge capacity c_{ij}.
  - Edge transportation cost b_{ij}.
- Player 1 (Defender) chooses a feasible flow $x \in \mathcal{F}$.
- Player 2 (Attacker) chooses the edges to disrupt through an attack $\mu \in \mathcal{A}$.

$$\forall (i,j) \in \mathcal{E},\ \mu_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is disrupted,} \\ 0 & \text{otherwise.} \end{cases}$$

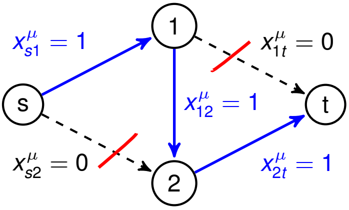
- 1 single $s - t$ pair.

Effective flow

- Given a flow x and an attack μ, x^{μ} is the **effective flow**.



Initial flow and attack.



Resulting effective flow

Payoffs

$\Gamma := \langle \{1,2\}, (\mathcal{F}, \mathcal{A}), (u_1, u_2) \rangle$

- $u_1(x, \mu) = p_1 \, \mathrm{F}(x^\mu) - \mathrm{C}_1(x)$
- $u_2(x, \mu) = p_2 \, \mathrm{F}(x - x^\mu) - \mathrm{C}_2(\mu)$

where:

- $\mathrm{F}(x^\mu) = \sum_{\{i \,|\, (i,t) \in \mathcal{E}\}} x^\mu_{it}$ is the **amount of effective flow**.
- $\mathrm{C}_1(x) = \sum_{(i,j) \in \mathcal{E}} b_{ij} x_{ij}$ is the **transportation cost**.
- $\mathrm{C}_2(\mu) = \sum_{(i,j) \in \mathcal{E}} c_{ij} \mu_{ij}$ is the **attacking cost**.
- $\mathrm{F}(x - x^\mu) = \mathrm{F}(x) - \mathrm{F}(x^\mu)$ is the **amount of lost flow**.

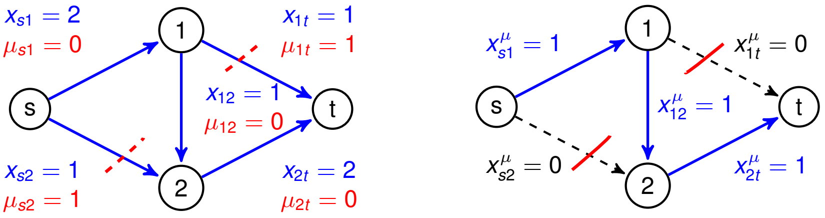
- Mixed-extension: for $(\sigma^1, \sigma^2) \in \Delta(\mathcal{F}) \times \Delta(\mathcal{A})$:

$$U_1(\sigma^1, \sigma^2) = \mathbb{E}[u_1(x, \mu)], \qquad U_2(\sigma^1, \sigma^2) = \mathbb{E}[u_2(x, \mu)]$$

- $\mathcal{S}_\Gamma$ is the set of Nash Equilibria.

Example

$\forall (i,j) \in \mathcal{E},\ b_{ij} = 1.$



Initial flow and attack. Resulting effective flow

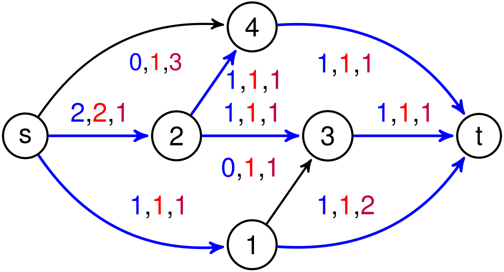
- $u_1(x,\mu) = p_1 - 7$
- $u_2(x,\mu) = 2p_2 - 2.$

What properties does $\mathcal{S}_\Gamma$ satisfy?

Simplification

Assumption 1

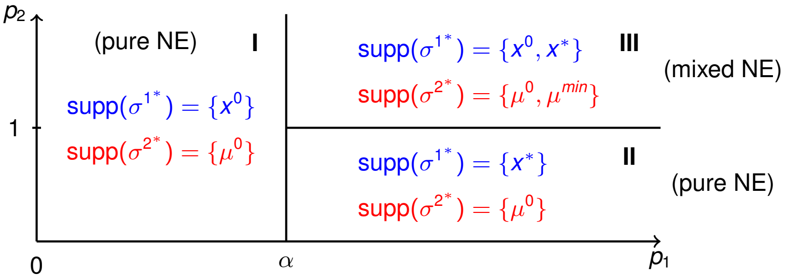
There exists a max-flow with min-transp. cost x^* that only takes $s - t$ paths that induce the lowest marginal transportation cost, denoted α.



- $\alpha = 3$

- Simplifying assumption without any loss of generality.
- α plays an important role in the results.

Regimes

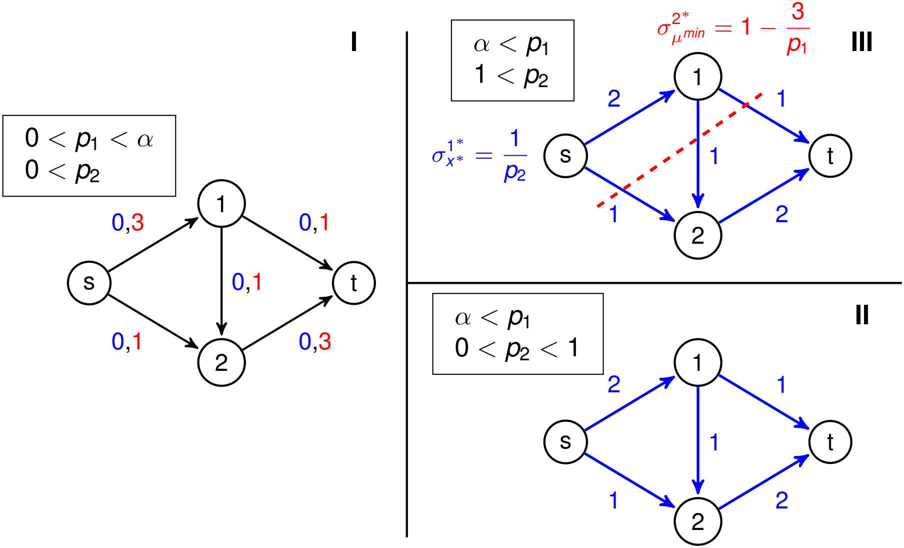


Proposition (Regime **III**)

If $p_1 > \alpha$ and $p_2 > 1$, then Γ has no pure NE. Furthermore, $\exists\, \sigma_0 = (\sigma_0^1, \sigma_0^2) \in \mathcal{S}_\Gamma$ such that $U_1(\sigma_0^1, \sigma_0^2) = U_2(\sigma_0^1, \sigma_0^2) = 0$. σ_0 is defined by:

- $\sigma^1_{x^0} = 1 - \dfrac{1}{p_2}, \quad \sigma^1_{x^*} = \dfrac{1}{p_2},$
- $\sigma^2_{\mu^0} = \dfrac{\alpha}{p_1}, \quad \sigma^2_{\mu^{min}} = 1 - \dfrac{\alpha}{p_1}$

- Example: Every path induces the same transportation cost



Necessary conditions

Probability bounds

Consider $(\sigma^{1^*}, \sigma^{2^*}) \in \mathcal{S}_\Gamma$. Then we have the following bounds:

- Player 1:
  - If $x^0 \in \text{supp}(\sigma^{1^*})$, then $\sigma^{1}_{x^0}{}^* \leq 1 - \dfrac{1}{p_2}$
  - If $x^* \in \text{supp}(\sigma^{1^*})$, then $\sigma^{1^*}_{x^*} \leq \dfrac{1}{p_2}$
- Player 2:
  - If $\mu^{min} \in \text{supp}(\sigma^{2^*})$, then $\sigma^{2^*}_{\mu^{min}} \leq 1 - \dfrac{\alpha}{p_1}$
  - If $\mu^0 \in \text{supp}(\sigma^{2^*})$, then $\sigma^{2^*}_{\mu^0} \leq \dfrac{\alpha}{p_1}$

- Remark: (σ^1_0, σ^2_0) proves that these bounds are tight.

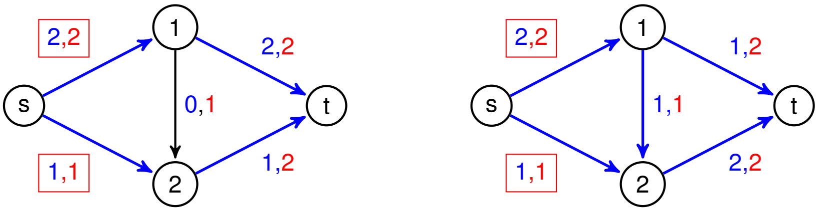
Necessary conditions

Attacker strategy σ^{2^*} and $(\mathcal{P}_2)$

For any NE $(\sigma^{1^*}, \sigma^{2^*})$, any μ in the support of σ^{2^*} disrupts edges that are saturated by every max-flow with minimum transportation cost.

$$\forall(\sigma^{1^*}, \sigma^{2^*}) \in \mathcal{S}_\Gamma,\ \forall \mu \in \mathrm{supp}(\sigma^{2^*}),\ \forall (i,j) \in \mathcal{E},\ \mu_{ij} = 1 \implies \forall x^* \in \Omega_2,\ x^*_{ij} = c_{ij}$$

Example: every path induces the same transportation cost.



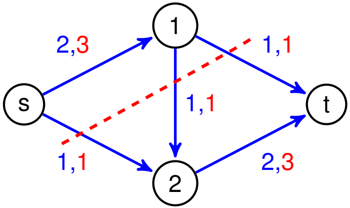
Necessary conditions

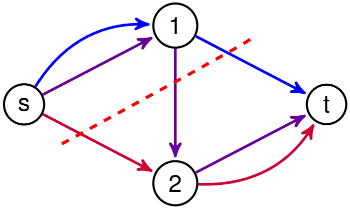
Defender strategy σ^{1^*} and min-cuts

For every NE $(\sigma^{1^*}, \sigma^{2^*})$, any edge of any min-cut must be taken by at least one flow x in the support of σ^{1^*}.

$$\forall(\sigma^{1^*}, \sigma^{2^*}) \in \mathcal{S}_\Gamma,\ \forall \text{ min-cut } E(\{S, T\}),\ \forall (i,j) \in E(\{S,T\}),$$
$$\exists\, x \in \text{supp}(\sigma^{1^*}) \mid x_{ij} > 0$$

Example:





Main results

$\Theta_1 = \mathrm{F}(x^*)$: Optimal value of the max-flow problem.
$\Theta_2 = \mathrm{C}_1(x^*)$: Optimal value of the max-flow min-cost problem.

Theorem (Regime **III**)

If $p_1 > \alpha$, $p_2 > 1$, and under Assumption 1, then for any $\sigma^* \in \mathcal{S}_\Gamma$:

1. *Both players' equilibrium payoffs are equal to 0, i.e.:*

$$U_1({\sigma^1}^*, {\sigma^2}^*) \equiv 0$$
$$U_2({\sigma^1}^*, {\sigma^2}^*) \equiv 0$$

2. *The expected amount of flow sent in the network is given by:*

$$\mathbb{E}_{\sigma^*}\left[\mathrm{F}(x)\right] \equiv \frac{1}{p_2}\Theta_1$$

and the expected transportation cost is given by:

$$\mathbb{E}_{\sigma^*}\left[\mathrm{C}_1(x)\right] \equiv \frac{1}{p_2}\Theta_2$$

Main results

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Theorem (Regime **III**)

3. *The expected cost of attack is given by:*

$$\mathbb{E}_{\sigma^*}\left[\mathrm{C}_2(\mu)\right] \equiv \Theta_1 - \frac{1}{p_1}\Theta_2 = \left(1 - \frac{\alpha}{p_1}\right)\Theta_1$$

4. *The expected amount of effective flow (that reaches t) is given by:*

$$\mathbb{E}_{\sigma^*}\left[\mathrm{F}(x^\mu)\right] \equiv \frac{1}{p_1 p_2}\Theta_2$$

5. *The yield is given by:*

$$\frac{\mathbb{E}_{\sigma^*}\left[\mathrm{F}(x^\mu)\right]}{\mathbb{E}_{\sigma^*}\left[\mathrm{F}(x)\right]} \equiv \frac{\Theta_2}{p_1\Theta_1}$$

- $\mathbb{E}_{\sigma^*}\left[\mathrm{F}(x^\mu)\right]$ decreases with p_1 and p_2!

Properties

Expected amount of edge flow in min-cuts

Consider a min-cut $E(\{S, T\})$, then:

$$\forall ({\sigma^1}^*, {\sigma^2}^*) \in \mathcal{S}_\Gamma, \ \forall (i,j) \in E(\{S, T\}), \ \mathbb{E}_{\sigma^*}[x_{ij}] = \frac{c_{ij}}{p_2}$$

Probability of edge disruption in min-cuts

For any NE whose support only contains attacks that disrupt edges of a single min-cut $E(\{S, T\})$, we have:

$$\forall (i,j) \in E(\{S, T\}), \ \mathbb{P}\left((i,j) \text{ is disrupted}\right) = 1 - \frac{\alpha}{p_1}.$$

Proof of the Theorem (outline)

Consider $\sigma^* = ({\sigma^1}^*, {\sigma^2}^*) \in \mathcal{S}_\Gamma$

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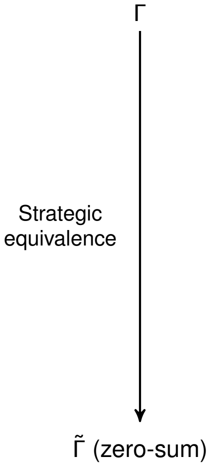

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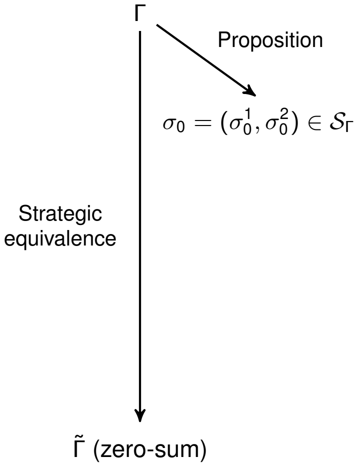

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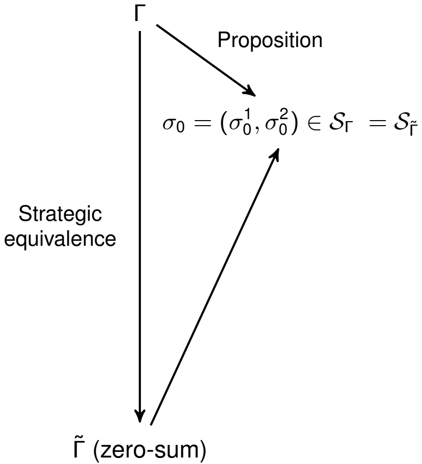

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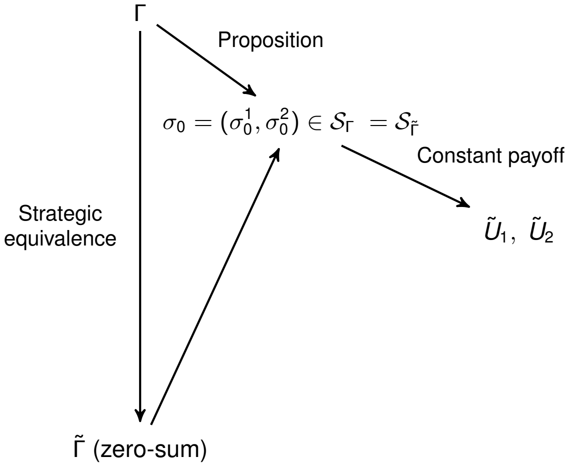

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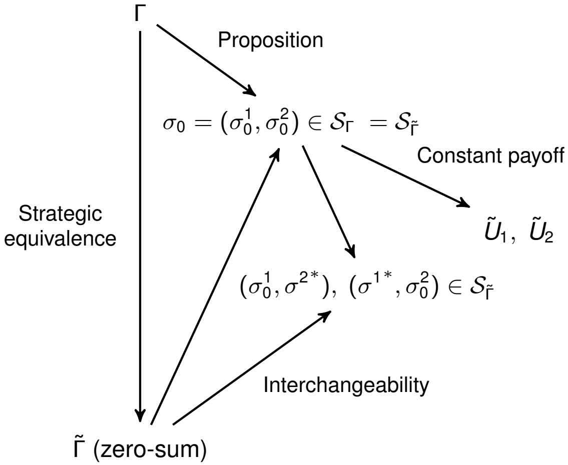
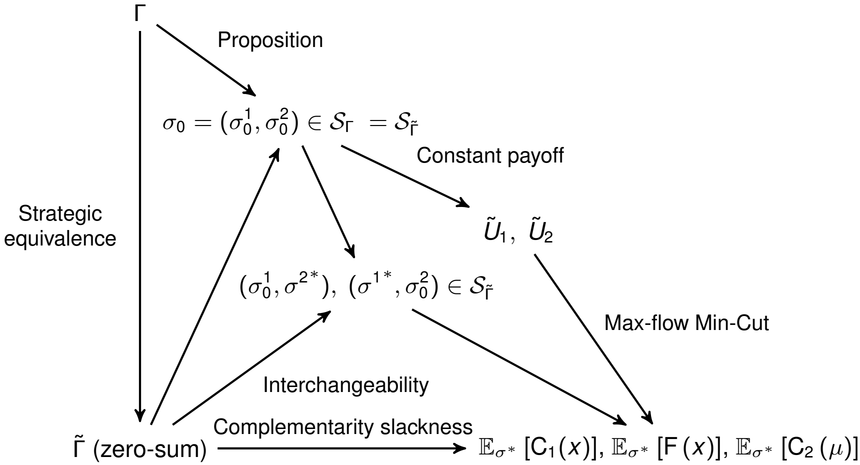


Proof of the Theorem (outline)

Consider $\sigma^* = (\sigma^{1*}, \sigma^{2*}) \in \mathcal{S}_\Gamma$



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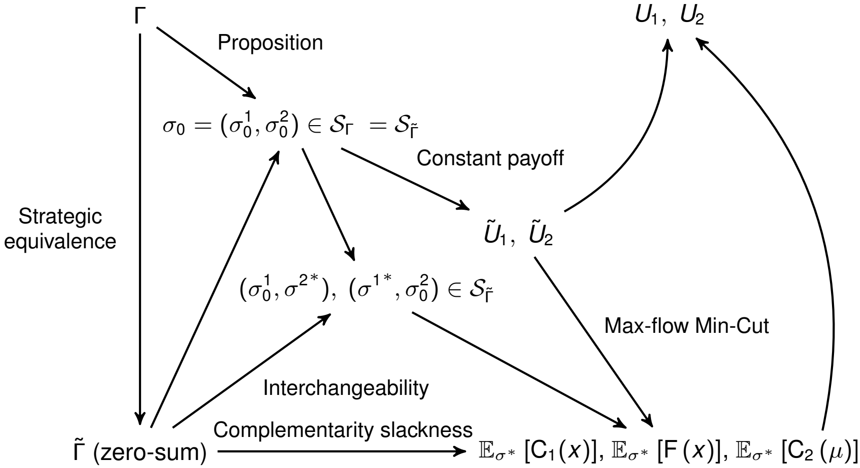

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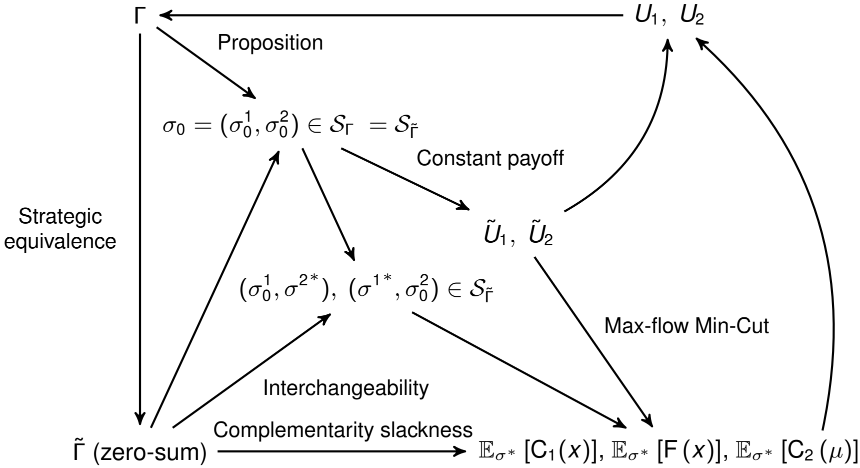
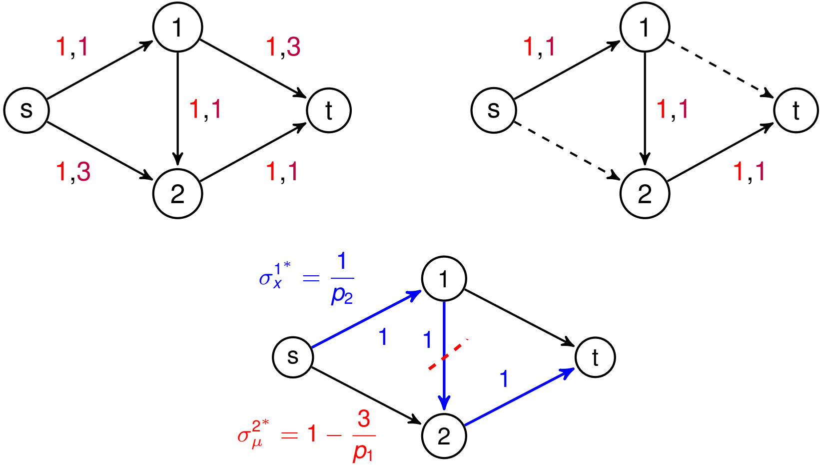


Relaxing Assumption 1

$3 < p_1 < 4$ and $p_2 > 1$.



- $\sigma^* = (\sigma^{1^*}, \sigma^{2^*})$ is a NE.

Conclusion and open questions

Results

- Modeled a simultaneous non-zero sum network game
- Obtained structural insights on the NE
- Related the NE to max-flow min-cost and min-cut
- Determined the vulnerability of a graph under strategic attack

Ongoing

- Nash equilibria of the one-stage game within the class of mixed strategies under link disruptions caused due to either reliability or security failures
- Equilibria for the finitely or infinitely repeated game

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Thank you!

Questions: mdahan@mit.edu, amins@mit.edu

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