



Distributed Online Learning in Multi-Agent Systems

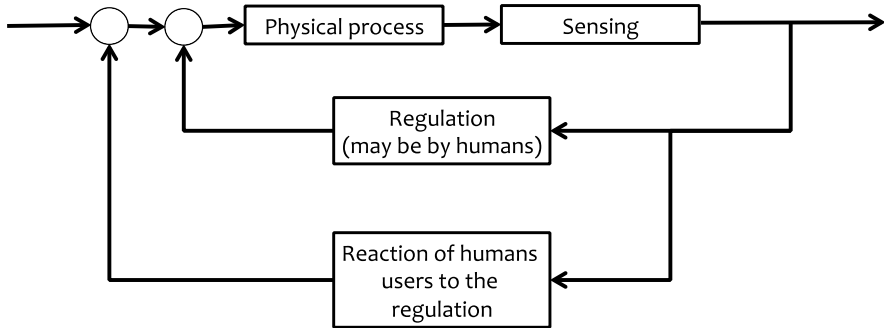
Alex Bayen

Dept. of Electrical Engineering & Computer Sciences,
UC Berkeley, CA, USA



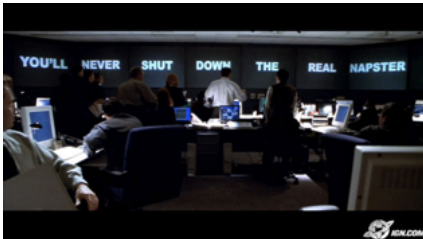
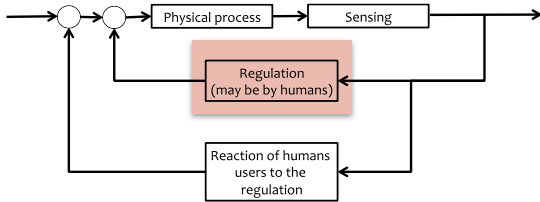
Project status within FORCES scope

Framework for cybersecurity of physical infrastructure



Roughly one attack a month on the traffic management infrastructure

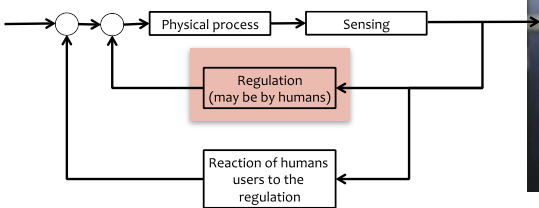
The *Italian Job* (2003)



Roughly one attack a month on the traffic management infrastructure

The *Italian Job* (2003)

The “real” *Italian Job* (2007)



Los Angeles Times

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Key signals targeted, officials say

Two accused of hacking into L.A.'s traffic light system plead not guilty. They allegedly chose intersections they knew would cause major jams.

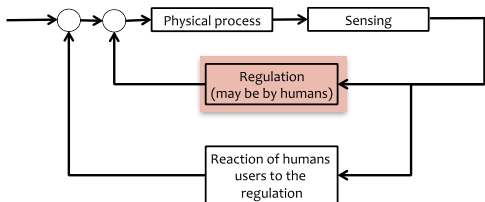
January 09, 2007 | Sharon Bernstein and Andrew Blankstein | Times Staff Writers

Roughly one attack a month on the traffic management infrastructure

The *Italian Job* (2003)

The “real” *Italian Job* (2007)

NC DOT signs hacked (2014)



FBI investigating hacked NCDOT digital road signs

Submitted by **WWAY** on Sat, 05/31/2014 - 9:55am.

READ MORE: News New Hanover County News Crime Cybercrime FBI Hacking N.C. NCDOT Transportation

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WILMINGTON, NC (WWAY) -- The North Carolina Department of Transportation says the FBI is looking into a group that hacked into at least five digital road signs yesterday, including one in New Hanover County.

The DOT says it is also evaluation the security measures in place for its digital road signs after a group changed the intended transportation-related messages on the signs to an advertisement for its Twitter account. According to a news released, the DOT corrected the messages as soon as it discovered the hackings.

The DOT says the hacked message boards are on Carolina Beach Road in New Hanover county, I-40 and I-240 in Asheville, US 421 in Winston-Salem and I-77 near the North Carolina/Virginia state line.



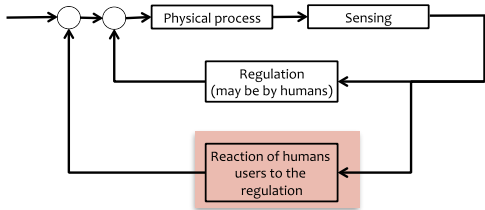
Roughly one attack a month on the traffic management infrastructure

The *Italian Job* (2003)

The “real” *Italian Job* (2007)

NC DOT signs hacked (2014)

Snail operations (2014)



The screenshot shows the Le Parisien website. The top navigation bar includes "ACTUALITÉS", "SPORTS", "MA VILLE", "CULTURE & LOISIRS", "VIDÉOS", "PHOTOS", "YOU", and "LA PARISIENNE". Below the navigation bar is a search bar with the text "Rechercher sur le site" and an "OK" button. The main content area features a "À SUIVRE" section with links to "La question du jour", "Grève SNCF", "France - Honduras", "Frondeurs du PS", and "Irak". Below this is a "À LA UNE" section with links to "SOCIÉTÉ", "FAITS DIVERS", "POLITIQUE", "COUPE DU MONDE", "ECONOMIE", "AUTO", and "INTERNATIONAL". The main article is titled "Manifestation des taxis ce matin : attention aux opérations escargot !" and is dated "Publié le 12.01.2014". Below the article title are social media sharing options for Facebook, Twitter, and Google+, along with a "Share" button.



Roughly one attack a month on the traffic management infrastructure

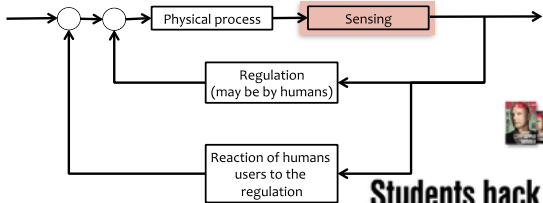
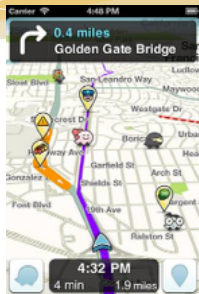
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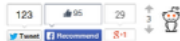
Snail operations (2014)

Waze / Google hacked (2014)



Students hack Waze, send in army of traffic bots

TECHNOLOGY / 25 MARCH 14 / by NICHOLAS TUFNELL



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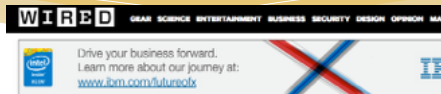
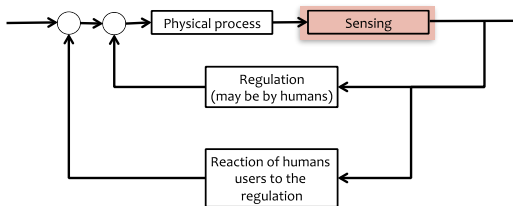
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Sensys Attack (2014)



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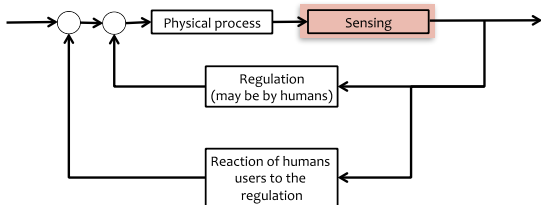
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Cesar Cerrudo in downtown New York City, conducting field test of vulnerable traffic sensors. Photo: Courtesy of Cesar Cerrudo

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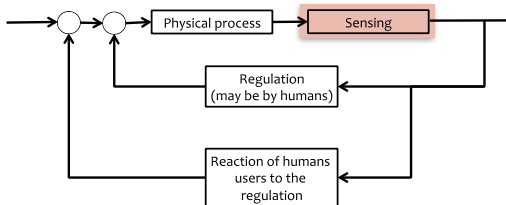
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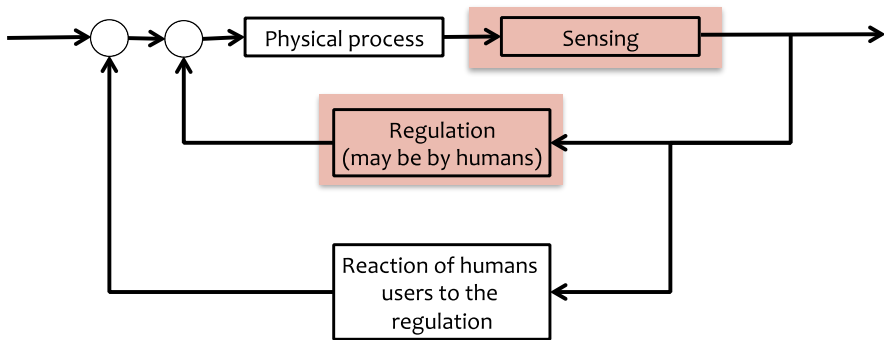
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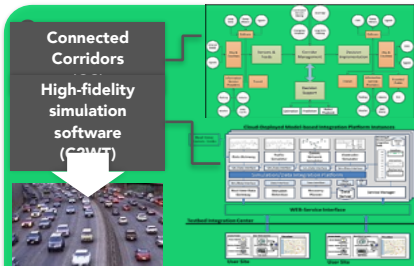
Presented at the previous FORCES all hands

Framework for cybersecurity of physical infrastructure



Presented at the previous FORCES all hands

Attack on the sensing infrastructure, and the control infrastructure

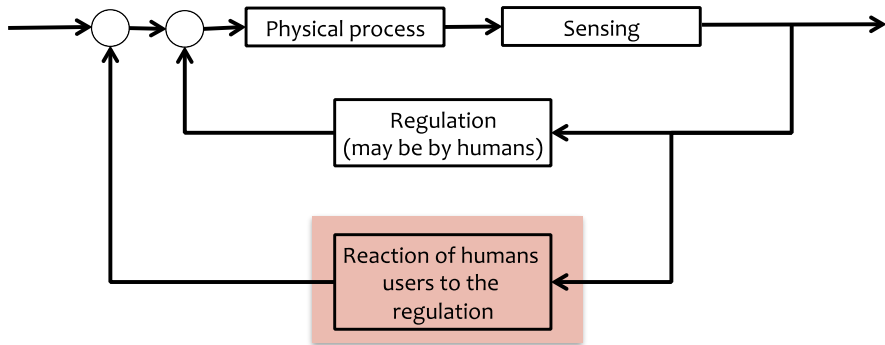


Well-managed and resilient traffic flows

This slide features the logos for Vanderbilt University and Berkeley University of California in the top left and right corners, respectively. The central focus is the SMARTROADS logo, which consists of a stylized road with a red and white striped border and a blue and white striped center, overlaid with a red 'X' and a blue circle. To the right of the logo, the word "SMARTROADS" is written in a bold, white, sans-serif font. Below the logo and text, the word "presents:" is written in a smaller, white font.

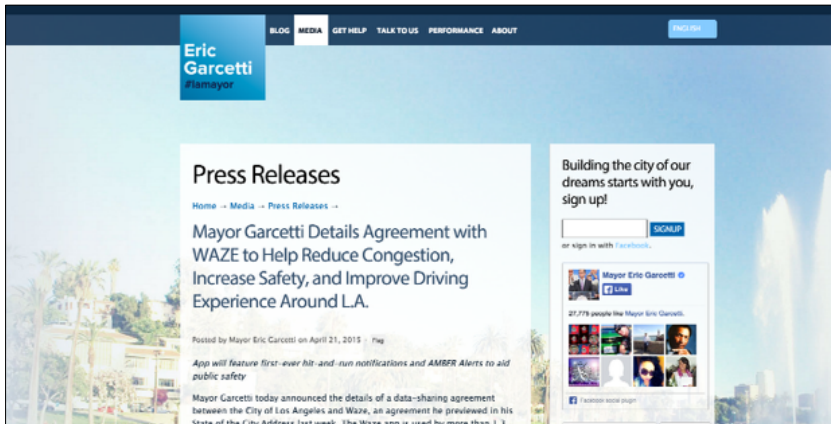
Presented at the this FORCES all hands

How do humans learn from attacks / ops / to protect / attack performance of the system



Context: (non-)collaborative players

Commonly spread ansatz about decision making in routing: if you reduce your own travel time, you help the public good.



The screenshot shows the website for Mayor Eric Garcetti. The header includes navigation links: BLOG, MEDIA, GET HELP, TALK TO US, PERFORMANCE, ABOUT, and a language selector for ENGLISH. The main content area features a "Press Releases" section with a headline: "Mayor Garcetti Details Agreement with WAZE to Help Reduce Congestion, Increase Safety, and Improve Driving Experience Around L.A." The text below the headline states: "App will feature first-ever hit-and-run notifications and AMBER Alerts to aid public safety." A sidebar on the right contains a sign-up form and a Facebook social widget for Mayor Eric Garcetti, showing 27,779 likes.

Context: (non-)collaborative players

Commonly spread ansatz about decision making in routing: if you reduce your own travel time, you help the public good.

The screenshot shows a news article on the Mobiquity website. The article title is "Los Angeles and Waze Team Up to Combat Traffic Congestion". The author is Eric Garcetti, Mayor of Los Angeles. The article text states that the city is partnering with the traffic app Waze to help combat congestion. The deal allows data to be shared between the two parties—the city will alert Waze about hazards, construction and crashes while the app will give the city a wealth of data to analyze how traffic moves. Ideally this will allow for changes that will improve commutes.

Eric Garcetti Mayor #lamayor

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mobiquity. make mobile matter

ABOUT HOW PORTFOLIO INSIGHTS CONTACT

Los Angeles and Waze Team Up to Combat Traffic Congestion

INSIGHTS | MOBILE DOSE

When Americans think of traffic they think of Los Angeles, even if they've never visited. So it makes sense that the LA mayor's office has announced that the city is **partnering with traffic app Waze** to help combat the congestion. The deal allows data to be shared between the two parties—the city will alert Waze about hazards, construction and crashes while the app will give the city a wealth of data to analyze how traffic moves. Ideally this will allow for changes that will improve commutes.

Mayor Garcetti today announced the details of a data-sharing agreement between the City of Los Angeles and Waze, an agreement he previewed in his State of the City Address last week. The Waze app is used by more than 1.3 billion people.

Context: (non-)collaborative players

Commonly spread ansatz about decision making in routing: if you reduce your own travel time, you help the public good.

The screenshot shows the top portion of The Boston Globe website. The navigation bar includes links for BLOG, MEDIA, GET HELP, TALK TO US, PERFORMANCE, and ABOUT, along with an ENGLISH button. The main header features the site's name, 'The Boston Globe', and a search icon. Below this, the 'beta Boston' logo is prominent. A section titled 'Today's top tech event' highlights a startup event on the 28th. The main article headline reads 'Boston partners with Google's Waze app to improve traffic flow in the city'. The article's lead image shows a highway at night with a 'STORM WARNING' sign and a smiley face sticker overlaid on the bottom right. A blue sidebar on the right contains the text 'Every day is a big day for Small Business.' and a subscription prompt for 'BetaBoston in your email' with an input field for an email address and a 'Daily' checkbox.

Page #:

Context: (non-)collaborative players

Reality: without optimal coordination, users experience the difference between social optimal and Nash

The screenshot shows a CNET news article. At the top, there is a red navigation bar with the CNET logo and search bar. Below the navigation bar, the article title is "Locals upset at Google's Waze for sending traffic to their streets". The sub-headline reads "LA residents complain that Waze creates congestion on roads once only known to those who live there." The byline is "By Dennis Taylor and @DavidTayl / December 14, 2014 9:26 AM PST". Below the byline are social media sharing icons for Facebook, Twitter, LinkedIn, StumbleUpon, and Email. A red advertisement banner for "rackspace" is visible, with the text "Tailor your cloud to your app. Not the other way around." Below the ad, there is a small image of a smartphone and a "THIS WEEK'S MUST READS" section with a link to "Locals upset at Google's Waze for sending traffic to their streets".

Context: (non-)collaborative players

Reality: without optimal coordination, users experience the difference between social optimal and Nash



Waze Has No Concept Of The Hell That Is LA Traffic



Brittany Malooly
4/22/15 2:18pm

652

Waze markets itself as a hip, modern, community-based app that helps urban drivers save time and stay safe on the road, but Waze is the very same company that is repeatedly fucking over Angelenos during rush hour traffic.

Ads by Google

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Waze consistently recommends something [people are referring to on Reddit](#) as the "suicide left," which entails turning from a small side street onto a busy, multi-lane road during peak traffic hours without a stoplight. Other users also complain that the app will suggest clearing the entire road straight across. Not only do these options waste time as drivers either wait for a chance to cross or turn, but these suggestions are also dangerous.

cnet

Intro
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Context: (non-)collaborative players

Reality: without optimal coordination, users experience the difference between social optimal and Nash

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Angry LA residents are trying to sabotage Waze data to stop side-street congestion

BY MICHAEL CARNEY ON NOVEMBER 17, 2014

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BY DENNIS KROHANE about an hour ago

Seymour Mersh and the dangers of corporate muckraking

BY MARK ARDS about 4 hours ago

More's yours of the stuff Chronicle

chance to cross or turn, but these suggestions are also dangerous.

Context: (non-)collaborative players

Reality: without optimal coordination, users experience the difference between social optimal and Nash

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'Cut-through' traffic caused by Waze app must stop, L.A. councilman says

POSTED BY JOHN SCHARREN ON APRIL 28, 2015 IN GOVERNMENT | 10688 VIEWS | 2 RESPONSES

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A Los Angeles city deal with traffic app Waze may be great, but some local communities are being inundated with "cut-through" traffic that must stop, a Los Angeles City Councilman said Tuesday.

Paul Krekorian introduced a motion to help local neighborhoods, saying Waze should send drivers away from residential streets and onto major roadways as part of the company's data sharing agreement with the city.

Mayor Eric Garcetti announced last week that the city is sharing road closure data with Waze to improve its service, and in return the city is getting the data to use for traffic management.

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\$20.8 Billion dollar question

The screenshot shows the SocialTimes website interface. At the top, there is a navigation bar with links for Facebook, Twitter, Internet, Infographics, and Gaming, along with a search icon. Below the navigation bar is an advertisement for the EGO Power+ mower, highlighting it as the industry's first 50V Li-Ion battery mower, which cuts up to 2 miles on a single charge and has a compact, foldable design. The main content area features the article title "Can Social Media Help To Reduce Traffic Congestion?" under the "INTERNET" category. The article is by kaihbutler, dated Jan. 7, 2011, at 4:00 PM. The article text begins with "We use social media to inform our friends about getting engaged. We use social media to tell our followers about a special event in town. Will we soon be using social media to warn other drivers of an accident?" and continues with "According to the experts, this is the future. Having social media available in cars will allow for more". To the right of the article is a sidebar with a subscription form titled "Get SocialTimes delivered straight to your inbox" and a "Send an anonymous tip" section. At the bottom of the sidebar is another advertisement for "MASTER'S BEER".

Annual cost of congestion: \$20.8B (TTI Urban Mob. Rep. 2012)

Problem set up: one shot / repeated game

N players are routing traffic

Private sector apps (Google, Waze, Apple, INRIX etc.)

Some public sector apps (511)

Except some specific public agencies, none of these players are solving for social optimal solutions.

At best, all of these are providing Nash solutions, i.e. routes in which each user has no incentive to change his/her trajectory.

What if companies “learned” from the past?

Problem set up: one shot / repeated game

Structure of a learning game:



- 1) Make choice (route)
- 2) Perform action (drive the route)
- 3) Compare outcome with external information
- 4) Learn (for next action)

Outline

- 1 Introduction
- 2 Convergence of agent dynamics
 - Background
 - Approximate replicator dynamics (AREP)
 - Distributed stochastic mirror descent dynamics (DSMD)
- 3 Application to routing

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Interaction of K decision makers

Decision maker k faces a sequential decision problem

At iteration t

- (1) chooses probability distribution $x_{\mathcal{A}_k}^{(t)}$ over action set \mathcal{A}_k
- (2) discovers a loss function $\ell_{\mathcal{A}_k}^{(t)} : \mathcal{A}_k \rightarrow [0, 1]$
- (3) updates distribution

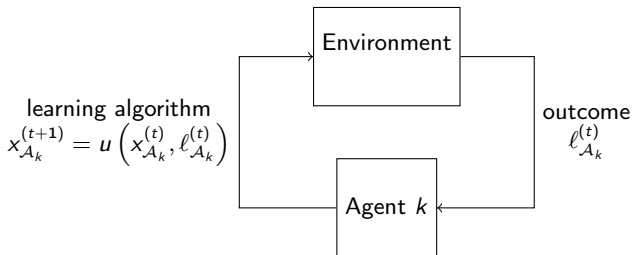


Figure: Sequential decision problem.

Loss of agent k affected by strategies of other agents.
Does not know this function, only observes its value.

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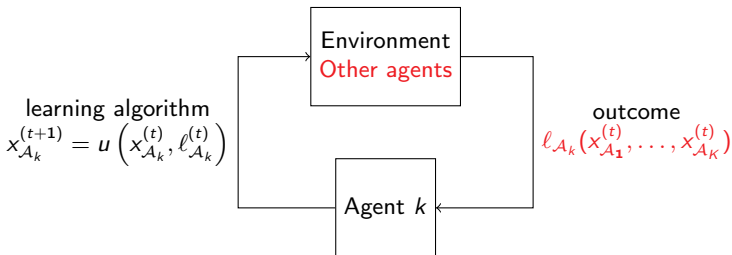


Figure: Sequential decision problem.

Loss of agent k affected by strategies of other agents.
Does not know this function, only observes its value.

Convergence to equilibria

- Can we guarantee $x^{(t)} \rightarrow \mathcal{X}^*$? Can players arrive to equilibrium?
 - $x^{(t)} = (x_{\mathcal{A}_1}^{(t)}, \dots, x_{\mathcal{A}_K}^{(t)})$
 - \mathcal{X}^* set of equilibria.
- Convergence rates?
- Robustness to stochastic perturbations?

Examples of decentralized decision makers

Routing game

- Player drives from source to destination node
- Chooses path from \mathcal{A}_k
- Mass of players on each edge determines cost on that edge.

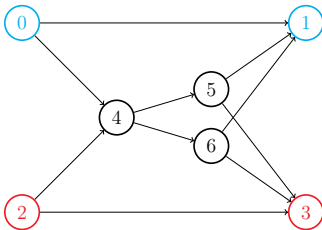


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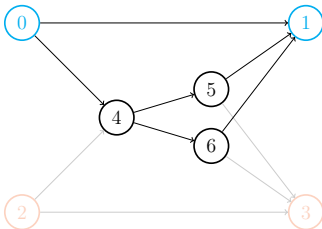
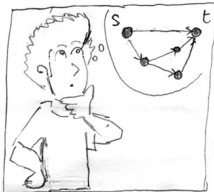


Figure: Routing game

Online learning model: illustration

$$x_{A_1}^{(t)} \in \Delta^{A_1}$$

Sample $a \sim x^{(t)}$ Discover $\ell_{A_1}^{(t)} \in [0, 1]^{A_1}$ Update $x_{A_1}^{(t+1)}$ 

Convergence of distributed learning

- 1: **for** $t \in \mathbb{N}$ **do**
- 2: Each agent k plays $x_{\mathcal{A}_k}^{(t)}$ (independently)
- 3: Reveal loss vector $\ell_{\mathcal{A}_k}(x^{(t)}) \in [0, 1]^{\mathcal{A}_k}$
- 4: Update

$$x_{\mathcal{A}_k}^{(t+1)} = u_k \left(x_{\mathcal{A}_k}^{(t)}, \ell_{\mathcal{A}_k}(x^{(t)}) \right)$$

- 5: **end for**

Main problem

Define class of dynamics \mathcal{C} such that

$$u_k \in \mathcal{C} \ \forall k \Rightarrow x^{(t)} \rightarrow \mathcal{X}^*$$

A brief review

Discrete time: uses regret analysis

- Hannan consistency: [5]
- Hedge algorithm for two-player games: [4]
- Online learning in games: [2]

Most studies prove the convergence of **time-averaged strategies**

$$\bar{x}^{(t)} = \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)}$$

[5] James Hannan. *Approximation to Bayes risk in repeated plays*. *Contributions to the Theory of Games*, 3:97–139, 1957

[4] Yoav Freund and Robert E Schapire. *Adaptive game playing using multiplicative weights*. *Games and Economic Behavior*, 29(1):79–103, 1999

[2] Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge University Press, 2006

Convergence of $\bar{x}(t)$ Vs convergence of $x(t)$

Routing game example

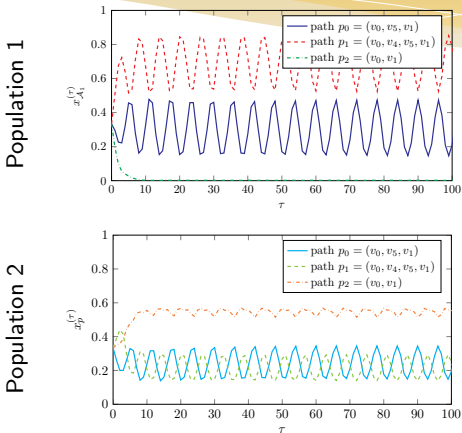
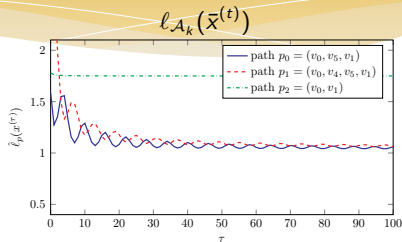
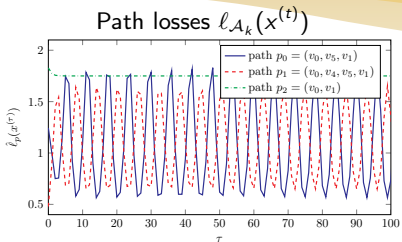


Figure: Population distributions

Convergence of $\bar{x}^{(t)}$ Vs convergence of $x^{(t)}$

Routing game example

Population 1



Population 2

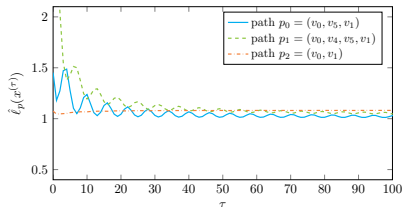
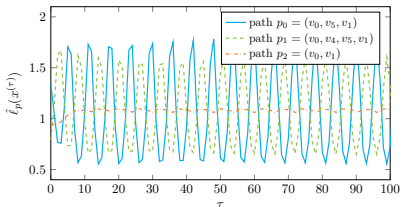


Figure: Path losses

Contributions

For the class of convex potential games.

Discrete time:

- $x^{(t)} \rightarrow \mathcal{X}^*$ for class of approximate replicator (AREP) dynamics [7]
- $x^{(t)} \rightarrow \mathcal{X}^*$ for class of distributed mirror descent (DMD) dynamics [9]
- $x^{(t)} \rightarrow \mathcal{X}^*$ for class of distributed stochastic mirror descent (DSMD) dynamics [6]

[7] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. [On the convergence of no-regret learning in selfish routing.](#)

In *31st International Conference on Machine Learning (ICML)*. JMLR, 2014

[9] Walid Krichene, Syrine Krichene, and Alexandre Bayen. [Convergence of mirror descent dynamics.](#)

In *European Control Conference (ECC), accepted*, 2015

[6] Syrine Krichene, Walid Krichene, Roy Dong, and Alexandre Bayen. [Convergence of stochastic mirror descent and applications to distributed routing.](#)

In *Allerton Conference on Communication, Control and Computing, in preparation*, 2015

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Convex potential

Convex potential

Assume that $\exists f$ convex on $\mathcal{X} = \Delta^{\mathcal{A}_1} \times \dots \times \Delta^{\mathcal{A}_K}$ such that

$$\nabla_{x_{\mathcal{A}_k}} f(x) = l_{\mathcal{A}_k}(x)$$

Then

$$\mathcal{X}^* = \arg \min_{x \in \Delta^{\mathcal{A}_1} \times \dots \times \Delta^{\mathcal{A}_K}} f(x)$$

is the set of Nash equilibria.

Write $\ell(x) = \nabla f(x) = (l_{\mathcal{A}_1}(x), \dots, l_{\mathcal{A}_K}(x))$

First order optimality conditions

$$\begin{array}{ccc} \text{Nash condition} & \Leftrightarrow & \text{first order optimality} \\ \forall k, \forall x_{\mathcal{A}_k}, \langle \ell_{\mathcal{A}_k}(x^*), x_{\mathcal{A}_k}^* - x_{\mathcal{A}_k} \rangle \leq 0 & \Leftrightarrow & \forall x \in \mathcal{X}, \langle \ell(x^*), x^* - x \rangle \leq 0 \end{array}$$

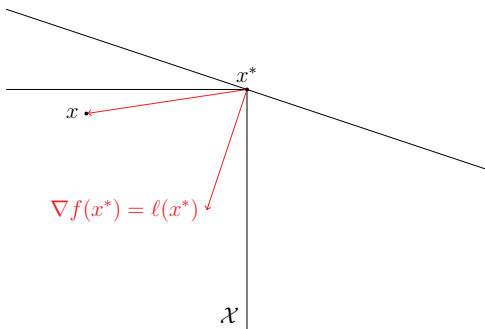


Figure: First order optimality conditions of the potential f

Convergence of $\bar{x}^{(t)}$

Cumulative regret

$$R_{\mathcal{A}_k}^{(t)} = \sup_{x_{\mathcal{A}_k} \in \Delta_{\mathcal{A}_k}} \sum_{\tau \leq t} \langle x_{\mathcal{A}_k}^{(\tau)} - x_{\mathcal{A}_k}, \ell_{\mathcal{A}_k}(x^{(\tau)}) \rangle$$

Convergence of averages

$$\left[\forall k, \limsup_t \frac{R_{\mathcal{A}_k}^{(t)}}{t} \leq 0 \right] \Rightarrow \bar{x}^{(t)} \rightarrow \mathcal{X}^*$$

Sufficient condition for $(x^{(t)})_t \rightarrow \mathcal{X}^*$

$f(x^{(t)})$ eventually decreasing

$$\Downarrow$$
$$f(x^{(t)}) \rightarrow f^*$$

$$\Downarrow$$
$$x^{(t)} \rightarrow \mathcal{X}^*$$

Convergence of $\bar{x}^{(t)}$

Cumulative regret

$$R_{\mathcal{A}_k}^{(t)} = \sup_{x_{\mathcal{A}_k} \in \Delta_{\mathcal{A}_k}} \sum_{\tau \leq t} \langle x_{\mathcal{A}_k}^{(\tau)} - x_{\mathcal{A}_k}, \ell_{\mathcal{A}_k}(x^{(\tau)}) \rangle$$

Convergence of averages

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Outline

- 1 Introduction
- 2 Convergence of agent dynamics
 - Background
 - **Approximate replicator dynamics (AREP)**
 - Distributed stochastic mirror descent dynamics (DSMD)
- 3 Application to routing

Replicator dynamics

Replicator equation [11]

$$\forall a \in \mathcal{A}_k, \frac{dx_a}{dt} = x_a (\langle \ell_{\mathcal{A}_k}(x), x_{\mathcal{A}_k} \rangle - \ell_a(x)) \quad (1)$$

Theorem: [3]

Every solution of the ODE (1) converges to the set of its stationary points.

[11] Jörgen W Weibull. *Evolutionary game theory*. MIT press, 1997

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Approximate REplicator update

Discretization of the continuous-time replicator dynamics

$$x_a^{(t+1)} - x_a^{(t)} = \eta_t x_a^{(t)} \left(\left\langle \ell_{\mathcal{A}_k}(x^{(t)}), x_{\mathcal{A}_k}^{(t)} \right\rangle - \ell_a(x^{(t)}) \right) + \eta_t U_a^{(t+1)}$$

- $(U^{(t)})_{t \geq 1}$ perturbations that satisfy for all $T > 0$,

$$\lim_{\tau_1 \rightarrow \infty} \max_{\tau_2: \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0$$

- η_t discretization time steps.

[1] Michel Benaïm. [Dynamics of stochastic approximation algorithms](#). In *Séminaire de probabilités XXXIII*, pages 1–68. Springer, 1999

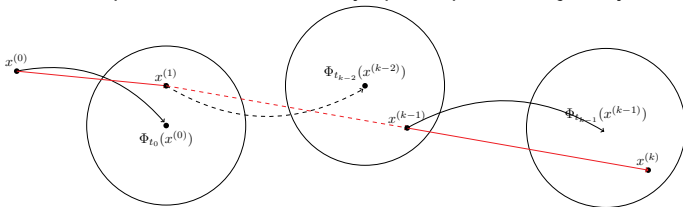
Convergence to Nash equilibria

Theorem [8]

Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x^{(t)} \rightarrow \mathcal{X}^*$$

- Affine interpolation of $x^{(t)}$ is an asymptotic pseudo trajectory.



- Use f as a Lyapunov function.

However, **No convergence rates.**

[8] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. Learning nash equilibria in congestion games.

SIAM Journal on Control and Optimization (SICON), to appear, 2014

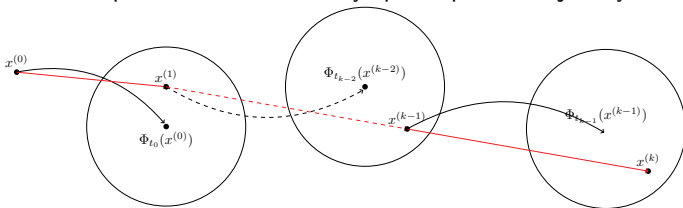
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Mirror Descent [10]

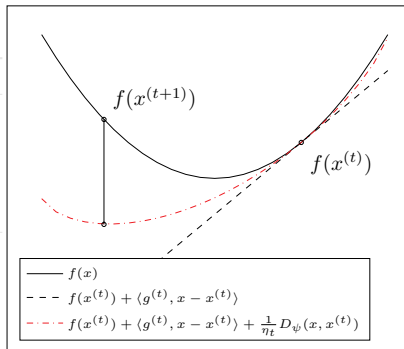
minimize $f(x)$ convex function
 subject to $x \in \mathcal{X} \subset \mathbb{R}^d$ convex, compact set

Algorithm 1 MD Method with learning rates (η_t)

- 1: for $t \in \mathbb{N}$ do
- 2: $g^{(t)} \in \partial f(x^{(t)})$
- 3: $x^{(t+1)} = \arg \min_{x \in \mathcal{X}} \langle g^{(t)}, x \rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)})$
- 4: end for

η_t : learning rate

D_{ψ} : Bregman divergence



[10] A. S. Nemirovsky and D. B. Yudin. *Problem complexity and method efficiency in optimization*.

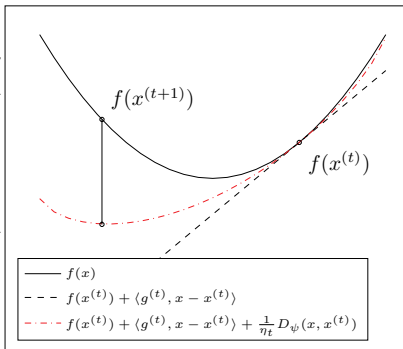
Wiley-Interscience series in discrete mathematics. Wiley, 1983

Mirror Descent [10]

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Algorithm 2 MD Method with learning rates (η_t)

- 1: **for** $t \in \mathbb{N}$ **do**
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Distributed Mirror Descent with heterogeneous agents

Distributed convex optimization

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && x \in \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K \end{aligned}$$

Algorithm 3 Distributed MD

- 1: **for** $t \in \mathbb{N}$ **do**
- 2: Agent k observes $g_{\mathcal{A}_k}^{(t)}$
- 3: Update

$$x_{\mathcal{A}_k}^{(t+1)} = \arg \min_{x_{\mathcal{A}_k} \in \mathcal{X}_k} \left\langle g_{\mathcal{A}_k}^{(t)}, x_{\mathcal{A}_k} - x_{\mathcal{A}_k}^{(t)} \right\rangle + \frac{1}{\eta_t^k} D_{\psi^k}(x_{\mathcal{A}_k}, x_{\mathcal{A}_k}^{(t)})$$

- 4: **end for**
-

- D_{ψ^k} and η_t^k depends on k

A true descent

Under mirror descent, $f(\bar{x}^{(t)}) \rightarrow f^*$.

A true descent [9]

If f has L -Lipschitz gradient, and $\eta_t \downarrow 0$, then eventually,

$$f(x^{(t+1)}) \leq f(x^{(t)})$$

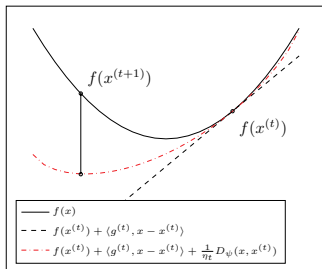


Figure: Mirror Descent iteration with decreasing η_t

[9] Walid Krichene, Syrine Krichene, and Alexandre Bayen. [Convergence of mirror descent dynamics.](#)

In *European Control Conference (ECC)*, accepted, 2015.

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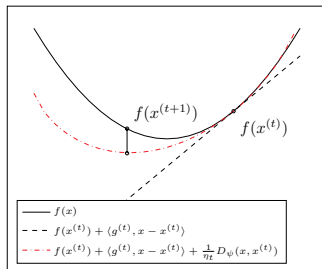


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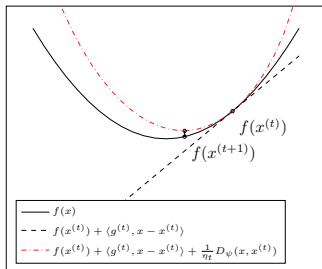


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A true descent

Consequence:

Theorem: Convergence of DMD [9]

Suppose f has L Lipschitz gradient. Then under the DMD class with $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$,

$$f(x^{(t)}) - f^* = O\left(\frac{\sum_{\tau \leq t} \eta_\tau}{t} + \frac{1}{t\eta_t} + \frac{1}{t}\right)$$

How robust is the convergence if losses are stochastic?

Players do not observe the true loss, but have an estimate $\hat{\ell}(x^{(t)})$

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Distributed Stochastic Mirror Descent (DSMD)

At iteration t

- Have a stochastic vector $\hat{g}^{(t)}$
- $\hat{g}^{(t)}$ unbiased: $\mathbb{E} \left[\hat{g}^{(t)} | \mathcal{F}_{t-1} \right] \in \partial f(x^{(t)})$ a.s.
(\mathcal{F}_t natural filtration of $(\hat{g}^{(t)})$)

Algorithm 4 DSMD dynamics

- 1: for $t \in \mathbb{N}$ do
- 2: Agent k observes $\hat{g}_{\mathcal{A}_k}^{(t)}$
- 3: Update

$$x_{\mathcal{A}_k}^{(t+1)} = \arg \min_{x_{\mathcal{A}_k} \in \mathcal{X}_k} \left\langle \hat{g}_{\mathcal{A}_k}^{(t)}, x_{\mathcal{A}_k} - x_{\mathcal{A}_k}^{(t)} \right\rangle + \frac{1}{\eta_t^k} D_{\psi^k}(x_{\mathcal{A}_k}, x_{\mathcal{A}_k}^{(t)})$$

- 4: end for
-

Assume

- $\exists G > 0$ s.t. $\mathbb{E} \left[\|\hat{g}^{(t)}\|_*^2 \right] \leq G \forall t$

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- 1: **for** $t \in \mathbb{N}$ **do**
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- 4: **end for**
-

Assume

- $\exists G > 0$ s.t. $\mathbb{E} \left[\|\hat{g}^{(t)}\|_*^2 \right] \leq G \forall t$

Convergence of DSMD

Existing result: $\mathbb{E}[f(\bar{x}^{(t)})] \rightarrow f^*$

Our results

- All convex functions (including non-smooth)

$$x^{(t)} \xrightarrow{a.s.} \mathcal{X}^*$$

$$\mathbb{E} \left[f(x^{(t)}) \right] - f^* = O \left(\sum_k \frac{\log t}{t^{\min(\alpha_k, 1 - \alpha_k)}} \right)$$

- Strongly convex functions

$$\mathbb{E} \left[D_{\psi}(x^*, x^{(t)}) \right] = O \left(\sum_k t^{-\alpha_k} \right)$$

Rates are for $\eta_t^k = \frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$

[6] Syrine Krichene, Walid Krichene, Roy Dong, and Alexandre Bayen. [Convergence of stochastic mirror descent and applications to distributed routing](#).

In *Allerton Conference on Communication, Control and Computing*, in preparation, 2015

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Application to the routing game

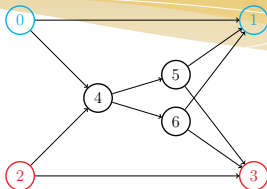


Figure: A strongly convex example.

- Centered Gaussian noise on edges.
- Population 1: Hedge with $\eta_t^1 = t^{-1}$
- Population 2: Hedge with $\eta_t^2 = t^{-1}$

Hedge algorithm

$$x_{\mathcal{A}_k}^{(t+1)} \propto x_a^{(t)} e^{-\eta_t \ell_a^{(t)}}$$

Routing game with strongly convex potential

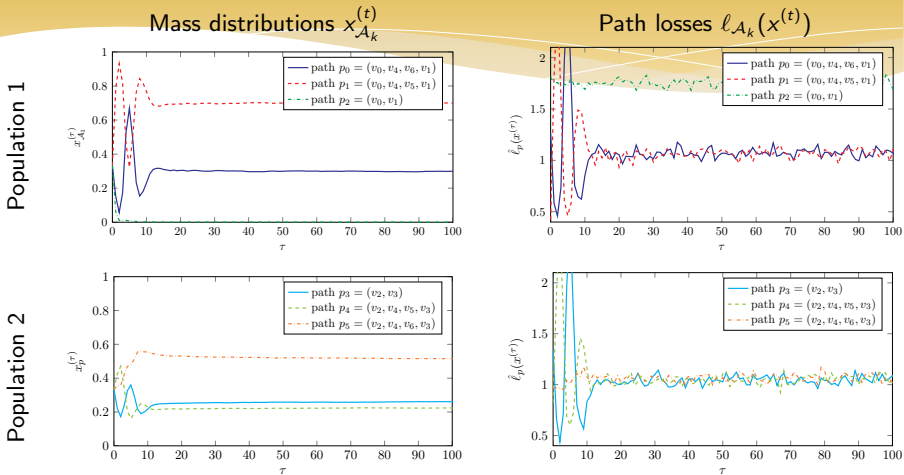
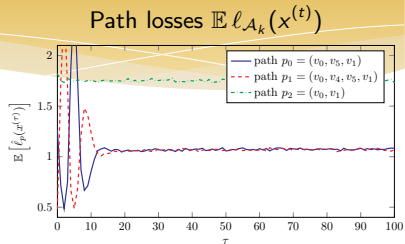
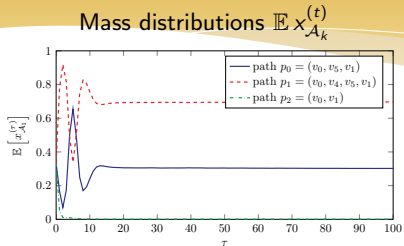


Figure: Population distributions and noisy path losses

Routing game with strongly convex potential

Population 1



Population 2

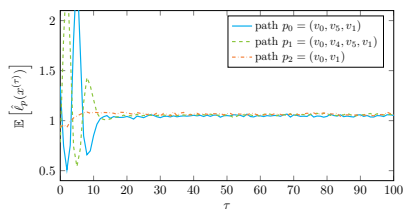
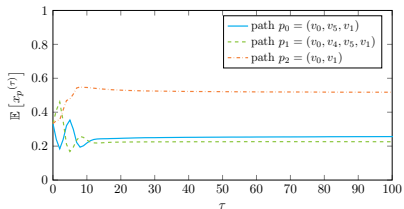


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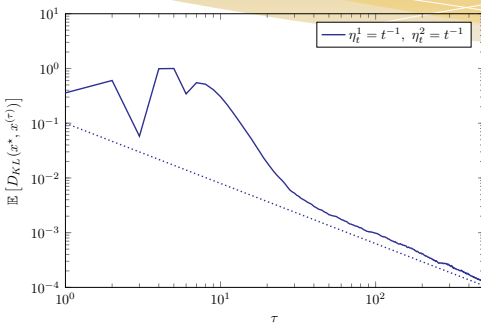


Figure: Distance to equilibrium.

For $\eta_t^k = \frac{\theta_k}{\ell_f t^{\alpha_k}}$, $\alpha_k \in (0, 1]$, $\mathbb{E}[D_\psi(x^*, x^{(t)})] = O(\sum_k t^{-\alpha_k})$

Routing game with weakly convex potential

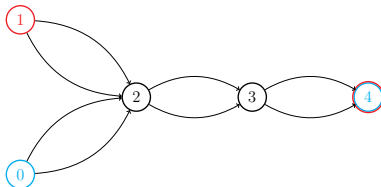


Figure: A weakly convex example.

Routing game with weakly convex potential

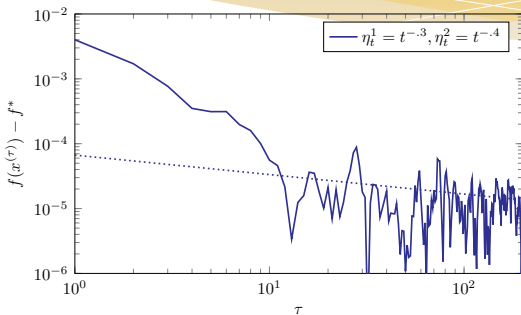


Figure: Potential values.

For $\frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$, $\mathbb{E} [f(x(t))] - f^* = O\left(\sum_k \frac{\log t}{t^{\min(\alpha_k, 1-\alpha_k)}}\right)$

Routing game with weakly convex potential

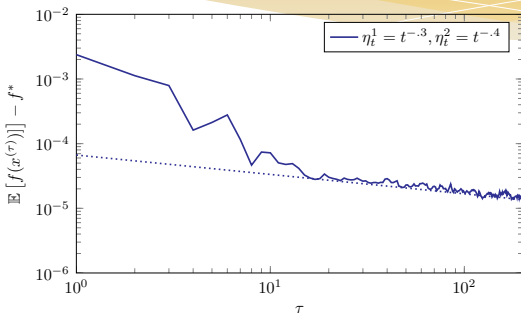


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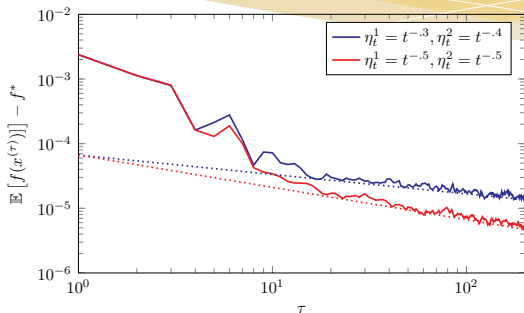


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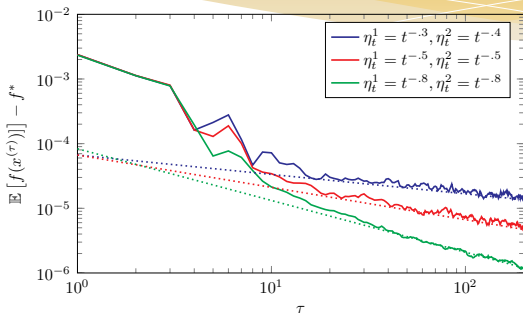


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Summary

Convergence guarantees for distributed dynamics

- Under no-regret learning, $\bar{x}^{(t)} \rightarrow \mathcal{N}$
- Under AREP dynamics, $x^{(t)} \xrightarrow{a.s.} \mathcal{N}$
- Under DMD dynamics, $x^{(t)} \rightarrow \mathcal{N}$ with rate $O\left(\frac{\sum_{\tau \leq t} \eta_\tau}{t} + \frac{1}{t\eta_t} + \frac{1}{t}\right)$
- Under Stochastic MD, $x^{(t)} \xrightarrow{a.s.} \mathcal{N}$, and $\mathbb{E} f(x^{(t)}) \rightarrow f^*$ with rate $O\left(\sum_k \frac{\log t}{t^{\min(\alpha_k, 1-\alpha_k)}}\right)$
- if potential is strongly convex, $\mathbb{E} D_\psi(x^*, x^{(t)}) \rightarrow 0$ with rate $O(\sum_k t^{-\alpha_k})$

Applications

- Distributed machine learning
- Used as a model for optimal control of potential games

Ongoing and future work

- Efficient Bregman projections (CDC 2015)
- Learning on a Continuum (NIPS 2015)
- Fitting of the learning model to observed dynamics

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References I

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- [3] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004.
- [4] Yoav Freund and Robert E Schapire. Adaptive game playing using multiplicative weights. *Games and Economic Behavior*, 29(1):79–103, 1999.
- [5] James Hannan. Approximation to Bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–139, 1957.
- [6] Syrine Krichene, Walid Krichene, Roy Dong, and Alexandre Bayen. Convergence of stochastic mirror descent and applications to distributed routing. In *Allerton Conference on Communication, Control and Computing, in preparation*, 2015.

References II

- [7] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *31st International Conference on Machine Learning (ICML)*. JMLR, 2014.
- [8] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. Learning nash equilibria in congestion games. *SIAM Journal on Control and Optimization (SICON)*, to appear, 2014.
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