NSF-CPS 1543830 An Entropy Framework for Comm. and Contr. In CPS

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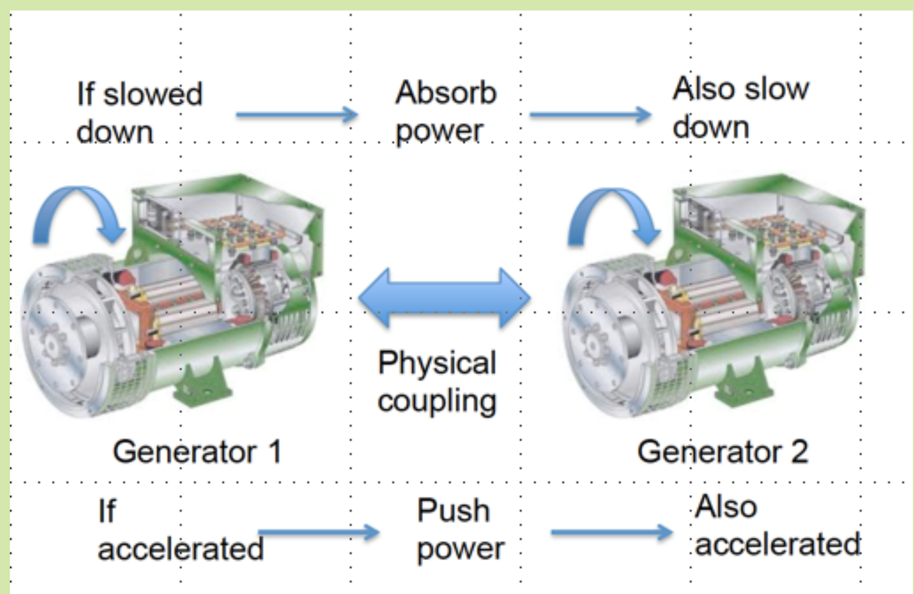
Networked Systems

- Due to the physical coupling in the networked CPSs, the entropy will be propagated in the networked physical dynamics nodes.
- Thus the communication requirements will be propagated.



System Model

- We consider N nodes with coupled physical dynamics in the cyber physical system (e.g., N generators in a power grid). These nodes form a network, in which $I \sim j$ means that nodes i and j are adjacent.
- We consider continuous time dynamics. The evolution of the system state of node i, $x_i(t)$, is given by

$$\dot{\mathbf{x}}_i(t) = \underbrace{\mathbf{A}_i\mathbf{x}_i(t)}_{\text{Self impact}} + \underbrace{\sum_{m\sim i}\mathbf{A}_{mi}\mathbf{x}_m(t)}_{\text{Impact of neighbors}} + \underbrace{\mathbf{B}_i\mathbf{u}_i(t)}_{\text{Control action}} + \underbrace{\mathbf{n}_i(t)}_{\text{Random perturbation}}.$$

- The differential entropy of the state of node i at time t is then defined as

$$h_i(t) = -\int \log(\rho_i(\mathbf{x}, t))\rho_i(\mathbf{x}, t)d\mathbf{x}.$$

where ρ_i is the probability density function of $x_i(t)$.

Entropy propagation: General distributions

Assumption

We first assume that the communication is perfect and consider only the entropy propagation. Then, the overall system dynamics can be rewritten as $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \mathbf{n}(t)$, where $\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{BK}$.

Theorem

Assume that each x_i is one-dimensional and $n_i(t) = 0$. Then, for any node i, the change of its marginal entropy has the following decomposition form: $\dot{h}_i(t) = E_{ii}(t) + \sum_{j\sim i} E_{ji}(t)$, where $E_{ii}(t)$ is the local entropy generation rate given by

$$E_{ii}(t) = -\tilde{A}_{ii}h_i(t) + \tilde{A}_{ii}\int x_i \log \rho_i(x_i, t)\frac{\partial \rho_i(x_i, t)}{\partial x_i}dx_i,$$

and $E_{ji}(t)$ is the entropy generation at node i caused by neighbor node j, which is given by

$$E_{ji}(t) = \tilde{A}_{ji}\int \log \rho_i(x_i, t)\frac{\partial\left(E(x_j|x_i)\rho_i(x_i, t)\right)}{\partial x_i}dx_i.$$

Entropy propagation: Gaussian Case

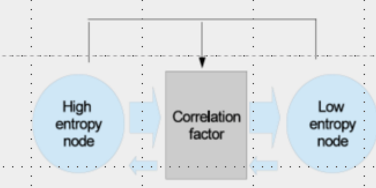
Corollary

When $\mathbf{x}$ is Gaussian distributed, we have

$$\begin{cases} E_{ii}(t) = \tilde{A}_{ii} \\ E_{ji}(t) = \tilde{A}_{ji}r_{ij}(t)e^{h_j(t)-h_i(t)} \end{cases}$$

where $r_{ij} = \frac{E[x_ix_j]}{\sqrt{E[x_i^2]E[x_j^2]}}$ is the correlation coefficient of x_i and x_j.

Intuition



- Once A_{ji}, A_{ij} and r_{ij} have the same signs, the entropy transfers between nodes i and j are both positive. Hence, the entropy may not be a conserved quantity in the network.
- We notice that the entropy transfer rate is dependent on both the correlation factor r_{ij} and entropy difference $h_j(t) - h_i(t)$. Hence, we can consider the entropy difference as the pressure and the correlation factor as a valve.

Interdependency with communications

Assumption

- Now we consider imperfect communications with limited capacity, which incurs quantization noise in the system state feedback.
- For the case of scalar system state, the variance of the quantization error e_j when estimating x_j at node i, denoted by σ_{ji}^e, is determined by the variance of $x_j(t)$ and the communication rate R_{ji}; i.e.,

$$\sigma_{ji}^e(t) = \sigma_j^2(t)e^{-2R_{ji}(t)}.$$

Theorem

We assume that the time period for sampling and communication is ΔT. For the case of scalar system states and Gaussian system state estimation noise, the entropy evolution law is given by

$$\begin{aligned} \dot{h}_i(t) &= E_{ii}(t) + \sum_{j\sim i} E_{ji}(t) \\ &+ \frac{1}{2}\sum_{j\sim i}(B_iK_{ji})^2 e^{-2(h_i(t)-h_j(t)+R_{ji}(t))}, \end{aligned}$$

as $\Delta T \to 0$.

Application

- The ODE consisting of both physical dynamics and communications can be used for jointing controlling the physical plant and communication resources.

Two types of dynamics

Consensus Dynamics (1st Order)

- Each node has a scalar system state, whose evolution law is given by the following first order differential equation:

$$\dot{x}_i(t) = A\sum_{j\sim i}(x_j(t) - x_i(t)),$$

where $A > 0$ is common for all nodes.
- It describes the consensus dynamics of the states of different nodes (e.g., the consensus control of voltages in microgrids).

Swing Dynamics (2nd Order)

- Each node has a 2-tuple system state $\mathbf{x}_i = (\theta_i, f_i)$, whose evolution law is given by the following differential equations:

$$\begin{cases} \dot{\theta}_i(t) &= f_i(t) \\ \dot{f}_i(t) &= c\sum_{j\sim i}(\theta_j(t) - \theta_i(t)) \end{cases}$$

where c is a constant common to all nodes.
- This describes the second order dynamics of electricity generators with small perturbation and negligible damping.

Continuum Limit of Consensus Dynamics

Theorem

Consider the planar consensus dynamics and assume $r = 1$. Then, as $\delta \to 0$, the evolution law of entropy $h(\mathbf{z}, t)$ converges to the following partial differential equation (PDE):

$$\frac{\partial h(\mathbf{z}, t)}{\partial t} = D\Delta h(\mathbf{z}, t) + D\left\|\nabla h(\mathbf{z}, t)\right\|^2,$$

where $D = \lim_{\delta\to 0} A\delta^2$, Δ is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and ∇ is the gradient operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$.

Remark

If we define $u(\mathbf{z}, t) = e^{h(\mathbf{z}, t)}$, where $u(\mathbf{z}, t) > 0$ (a.k.a. the Hopf-Cole transformation), then u satisfies the following heat (diffusion) equation $\frac{\partial u(\mathbf{z}, t)}{\partial t} = D\Delta u(\mathbf{z}, t)$. u is actually proportional to the standard deviation σ of system state x.

Swing Dynamics on Real Line: PDE Description

Variables

Recall that each node has state (θ, f), namely (phase, frequency). We study the evolution of covariances of phase and frequency, denoted by $\Sigma_{\theta\theta}$ and Σ_{ff}.

Theorem (PDE Description)

For the swing dynamics on the real line, as $\delta \to 0$, the dynamics of $\Sigma_{\theta\theta}(\mathbf{z}, t)$ and $\Sigma_{ff}(\mathbf{z}, t)$ are given by the following PDE:

$$\begin{cases} \frac{\partial^2\Sigma_{\theta\theta}(\mathbf{z})}{\partial t^2} = C\Delta\Sigma_{\theta\theta} + 2\Sigma_{ff} \\ \frac{\partial^2\Sigma_{ff}(\mathbf{z})}{\partial t^2} = C\Delta\Sigma_{ff} + 2C^2\frac{\partial^4}{\partial x^2\partial y^2}\Sigma_{\theta\theta} \end{cases}.$$

Corollary

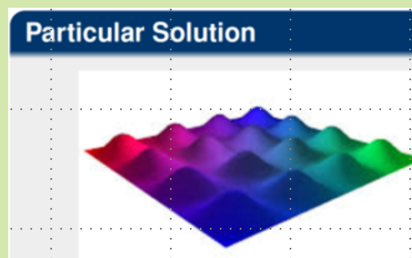
Both $\Sigma_{\theta\theta}$ and Σ_{ff} satisfy the following PDE:

$$\frac{\partial^4}{\partial t^4}u - 2C\frac{\partial^2}{\partial t^2}\Delta u + C^2\Box u = 0,$$

where $\Box = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)^2$.

Swing Dynamics on Real Line: Solution

Particular Solution



The following function (standing wave) is a particular solution:

$$\begin{aligned} u((x, y), t) &= \sin(k_x x + \theta_x) \\ &\times \sin(k_y y + \theta_y) \\ &\times \sin(wt + \theta_t). \end{aligned}$$

where $w = \pm(k_x + k_y)$ or $w = \pm(k_x - k_y)$, and k_x and k_y are real numbers. This implies the phenomenon of waves.

Theorem (General Solution)

Given the initial values $g_0(\mathbf{z})$, $g_1(\mathbf{z})$, $g_2(\mathbf{z})$ and $g_3(\mathbf{z})$, the general solution is given by

$$\begin{aligned} u(\mathbf{z}, t) &= \frac{1}{4}g_0(\mathbf{z} - \mathbf{v}_0t) + \frac{1}{4}g_0(\mathbf{z} - \mathbf{v}_1t) \\ &+ \frac{1}{4}g_0(\mathbf{z} - \mathbf{v}_2t) + \frac{1}{4}g_0(\mathbf{z} - \mathbf{v}_3t) + \oint_{\partial\Omega\mathbf{z}, t)}\mathbf{F}_0 \cdot d\mathbf{r} \\ &- \frac{1}{4\sqrt{C}}\int_{C\to A} g_1(\mathbf{z}(t))dt + \frac{1}{4\sqrt{C}}\int_{D\to B} g_1(\mathbf{z}(t))dt \\ &+ \frac{1}{8\sqrt{C}}\oint_{\partial\Omega} g_1(\mathbf{z}(t))dt + \frac{1}{8C}\int_{\Omega(\mathbf{z}, t)} g_2(\mathbf{z}')d\mathbf{z}' \\ &+ \int_{\sum_{i=0}^3 x_i = t, x_i \geq 0} g_3\left(\mathbf{z} - \sum_{i=0}^3 \mathbf{v}_i x_i\right)d\mathbf{x}. \end{aligned}$$

The physical meaning is still under exploration.