Logical Foundations of Cyber-Physical Systems

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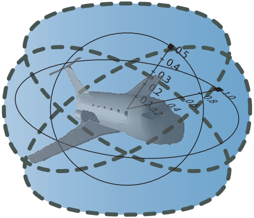
http://symbolaris.com/



Outline

1. CPS are Multi-Dynamical Systems
   - Hybrid Systems
   - Hybrid Games
   - Stochastic Hybrid Systems
   - Distributed Hybrid Systems
2. Dynamic Logic of Multi-Dynamical Systems
   - Syntax
   - Semantics
3. Proofs for CPS
4. Theory of CPS
   - Soundness and Completeness
   - Differential Invariants
   - Differential Radical Invariants
5. Applications
6. Summary

Can you trust a computer to control physics?

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Rationale

1. Safety guarantees require analytic foundations
2. Foundations revolutionized digital computer science & society
3. Need even stronger foundations when software reaches out into our physical world

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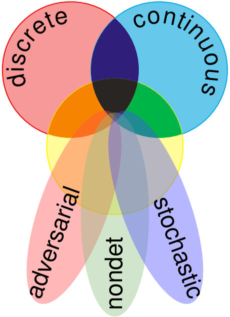
CPS Core Question

How can we provide people with cyber-physical systems they can bet their lives on?

CPS are Multi-Dynamical Systems

CPS Dynamics Bee

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combine multiple simple dynamical effects.

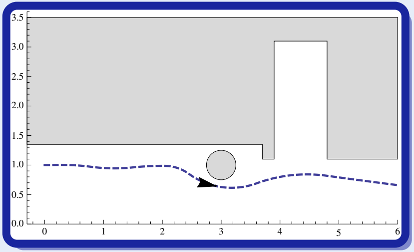
Tame Parts

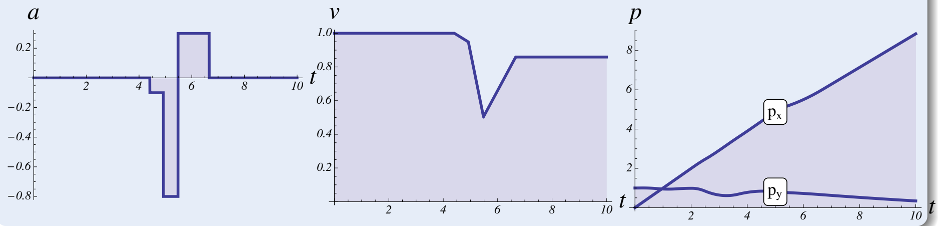
Exploiting compositionality tames complexity.

Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)





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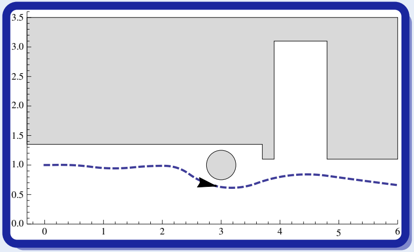

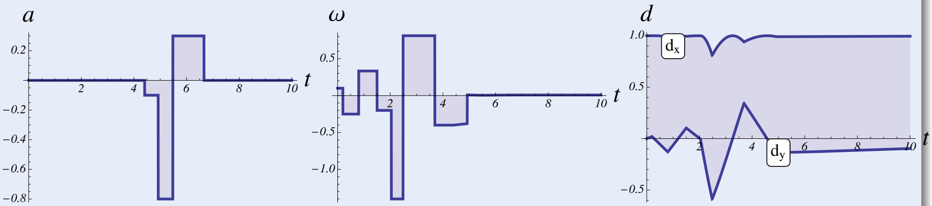

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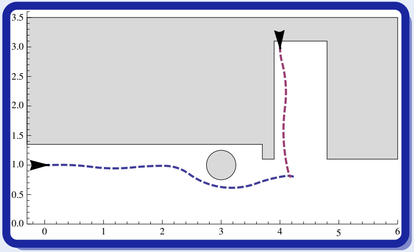

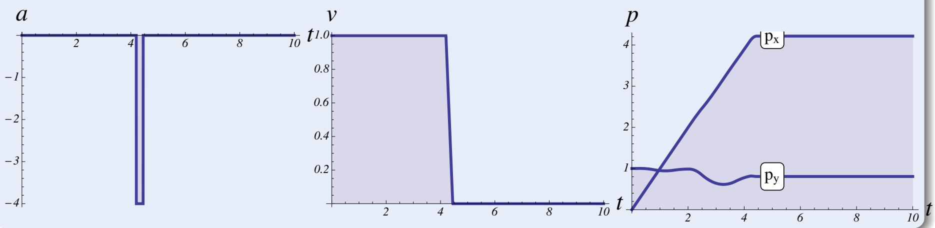

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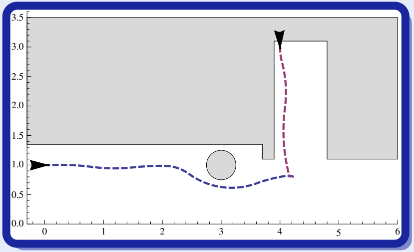
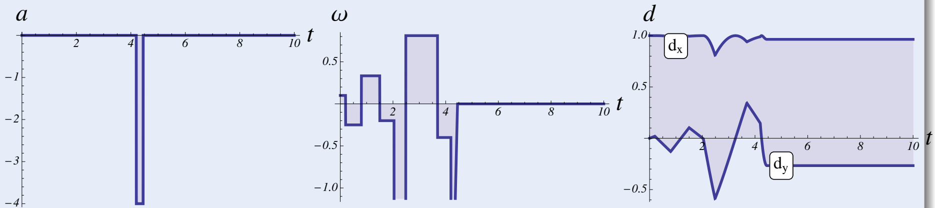
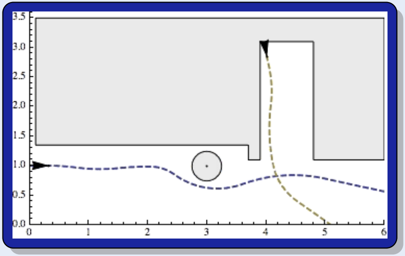


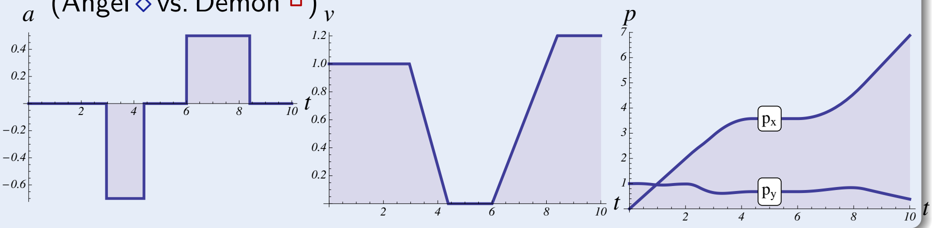


Challenge (Hybrid Games)

Game rules describing play evolution with

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)





CPS Analysis: Other Agents

Challenge (Hybrid Games)

Game rules describing play evolution with

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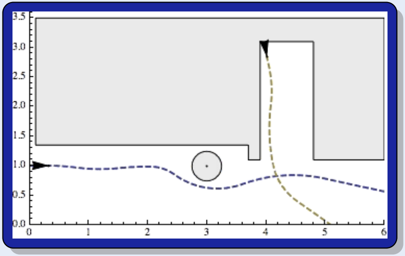
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel ◇ vs. Demon □)



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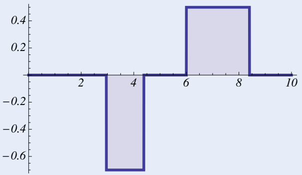

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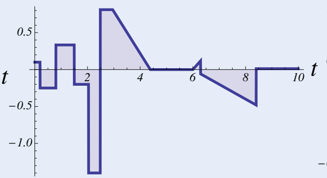

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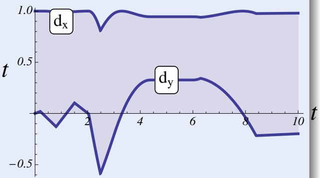
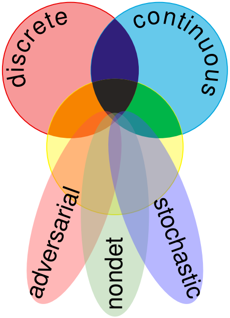


CPS are Multi-Dynamical Systems

hybrid systems

HS = discrete + ODE

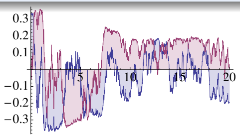


hybrid games

HG = HS + adversary

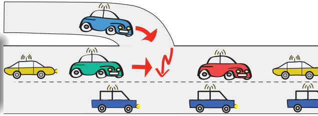
stochastic hybrid sys.

SHS = HS + stochastics

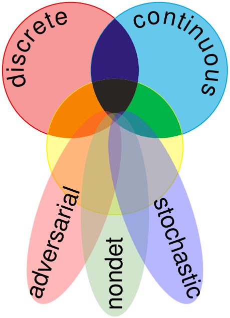


distributed hybrid sys.

DHS = HS + distributed



Family of Differential Dynamic Logics

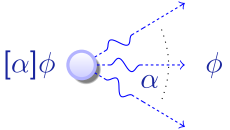


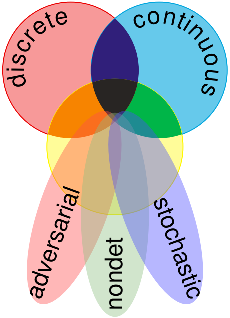
JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Family of Differential Dynamic Logics

differential dynamic logic

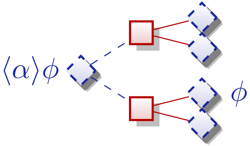
$\mathsf{d}\mathcal{L} = \mathsf{DL} + \mathsf{HP}$





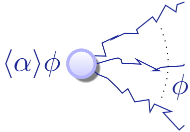
differential game logic

$\mathsf{dG}\mathcal{L} = \mathsf{GL} + \mathsf{HG}$



stochastic differential DL

$\mathsf{Sd}\mathcal{L} = \mathsf{DL} + \mathsf{SHP}$



quantified differential DL

$\mathsf{Qd}\mathcal{L} = \mathsf{FOL} + \mathsf{DL} + \mathsf{QHP}$

JAR'08,CADE'11,LMCS'12,LICS'12,LICS'12

Differential Dynamic Logic $d\mathcal{L}$: Syntax

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition ($d\mathcal{L}$ Formula ϕ)

$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\, \phi \mid \exists x\, \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

Differential Dynamic Logic d$\mathcal{L}$: Syntax

Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Definition (Hybrid program α)

$$x := \theta \mid ?H \mid x' = f(x) \,\&\, H \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

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$$\theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \forall x\,\phi \mid \exists x\,\phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$$

All Reals | Some Reals | All Runs | Some Runs

Differential Dynamic Logic $d\mathcal{L}$: Semantics

Definition (Hybrid program α)

$$
\begin{aligned}
\rho(x := \theta) &= \{(v, w) \ : \ w = v \text{ except } [\![x]\!]_w = [\![\theta]\!]_v\} \\
\rho(?H) &= \{(v, v) \ : \ v \models H\} \\
\rho(x' = f(x)) &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r\} \\
\rho(\alpha \cup \beta) &= \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha; \beta) &= \rho(\beta) \circ \rho(\alpha) \\
\rho(\alpha^*) &= \bigcup_{n \in \mathbb{N}} \rho(\alpha^n)
\end{aligned}
$$

Definition ($d\mathcal{L}$ Formula ϕ)

$$
\begin{aligned}
&v \models \theta_1 \geq \theta_2 && \text{iff} \quad [\![\theta_1]\!]_v \geq [\![\theta_2]\!]_v \\
&v \models [\alpha]\phi && \text{iff} \quad w \models \phi \text{ for all } w \text{ with } v\rho(\alpha)w \\
&v \models \langle\alpha\rangle\phi && \text{iff} \quad w \models \phi \text{ for some } w \text{ with } v\rho(\alpha)w \\
&v \models \forall x\, \phi && \text{iff} \quad w \models \phi \text{ for all } w \text{ that agree with } v \text{ except for } x \\
&v \models \exists x\, \phi && \text{iff} \quad w \models \phi \text{ for some } w \text{ that agrees with } v \text{ except for } x \\
&v \models \phi \wedge \psi && \text{iff} \quad v \models \phi \text{ and } v \models \psi
\end{aligned}
$$

Differential Dynamic Logic: Axiomatization

equations of truth

$[:=]$ $[x := \theta]\phi(x) \leftrightarrow \phi(\theta)$

$[?]$ $[?H]\phi \leftrightarrow (H \rightarrow \phi)$

$[']$ $[x' = f(x)]\phi \leftrightarrow \forall t{\geq}0\, [x := y(t)]\phi$ $\quad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \land [\beta]\phi$

$[;]$ $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$

$[^*]$ $[\alpha^*]\phi \leftrightarrow \phi \land [\alpha][\alpha^*]\phi$

K $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$

I $[\alpha^*](\phi \rightarrow [\alpha]\phi) \rightarrow (\phi \rightarrow [\alpha^*]\phi)$

C $[\alpha^*]\forall v{>}0\, (\varphi(v) \rightarrow \langle\alpha\rangle\varphi(v-1)) \rightarrow \forall v\, (\varphi(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v))$

LICS'12

Differential Dynamic Logic: Axiomatization

equations of truth

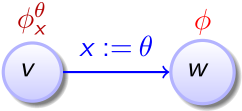
G $$\frac{\phi}{[\alpha]\phi}$$

MP $$\frac{\phi \rightarrow \psi \quad \phi}{\psi}$$

∀ $$\frac{\phi}{\forall x\, \phi}$$

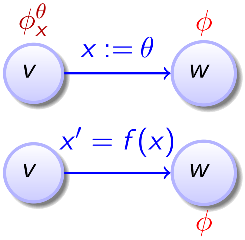
LICS'12

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$



$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

$$\frac{\forall t{\geq}0\, [x := y_x(t)]\phi}{[x' = f(x)]\phi}$$



Proofs for Hybrid Systems

$$\frac{\phi_x^\theta}{[x := \theta]\phi}$$

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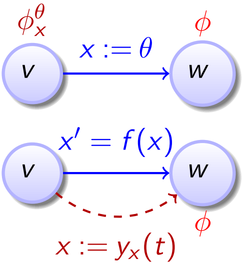
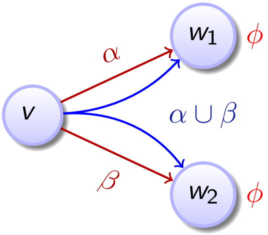


Proofs for Hybrid Systems

compositional semantics $\Rightarrow$ compositional rules!

Proofs for Hybrid Systems

$$\frac{[\alpha]\phi \land [\beta]\phi}{[\alpha \cup \beta]\phi}$$



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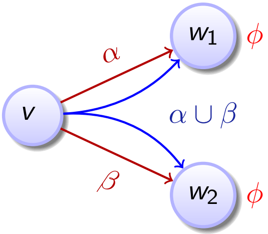
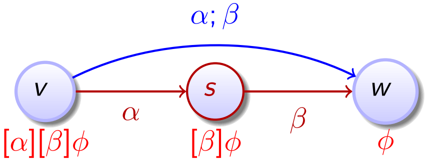


$$\frac{[\alpha][\beta]\phi}{[\alpha; \beta]\phi}$$



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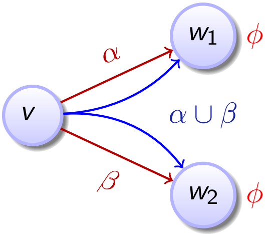

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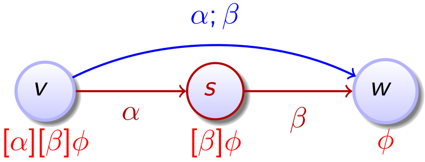
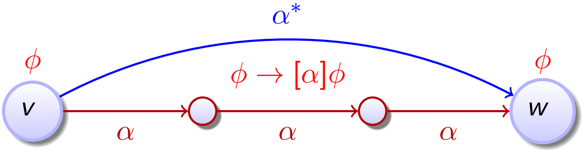


$$\frac{\phi \quad (\phi \to [\alpha]\phi)}{[\alpha^*]\phi}$$



Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (J.Autom.Reas. 2008, LICS'12)

$d\mathcal{L}$ *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* **or** *discrete dynamics.* Proof 25pp

Corollary (Complete Proof-theoretical Alignment & Bridging)

proving continuous = proving hybrid = proving discrete

JAutomReas'08,LICS'12

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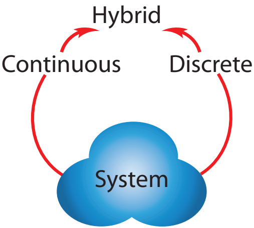

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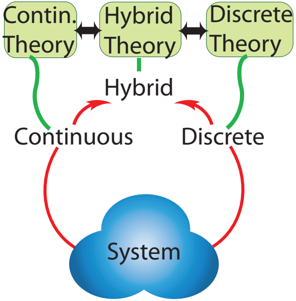
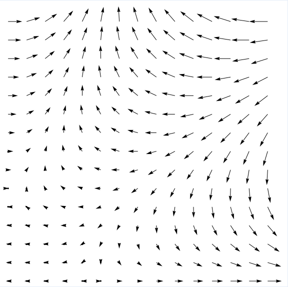


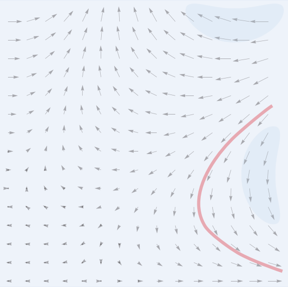
JAutomReas'08,LICS'12

Differential Invariants for Differential Equations

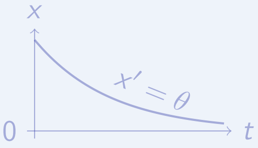
Differential Invariant

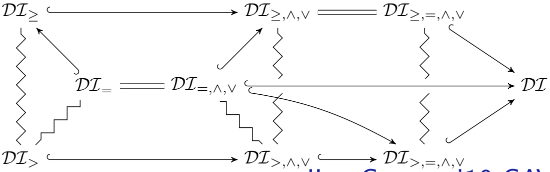


Differential Cut



Differential Ghost





Logic

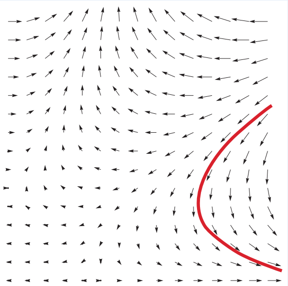
Provability study

Math

Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariant



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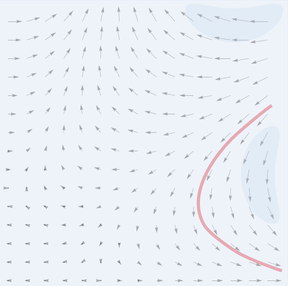
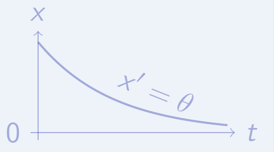
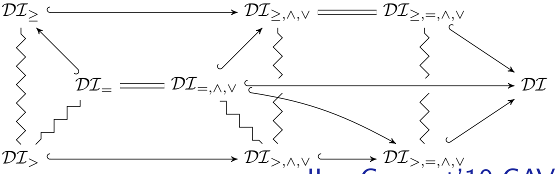


Differential Ghost





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JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

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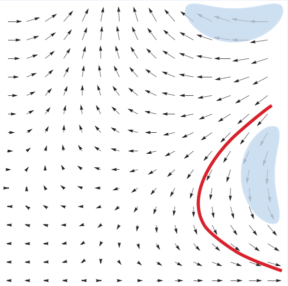

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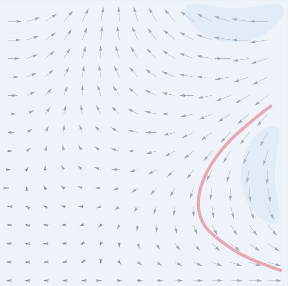

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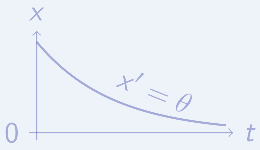
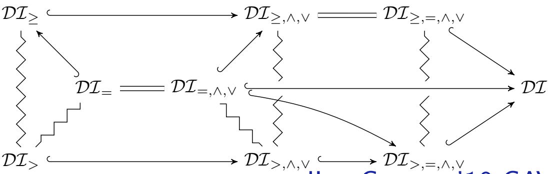




Logic

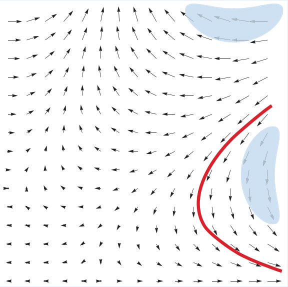
Provability study

Math

Character-istic PDE

JLogComput'10, CAV'08, FMSD'09, LMCS'12, ITP'12

Differential Invariant



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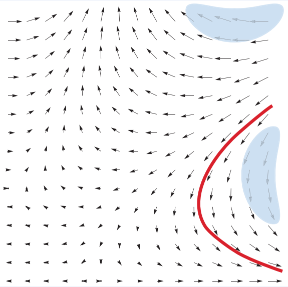
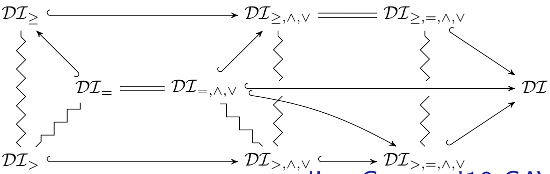


Differential Ghost



Logic

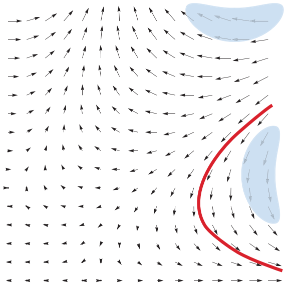
Provability study

Math

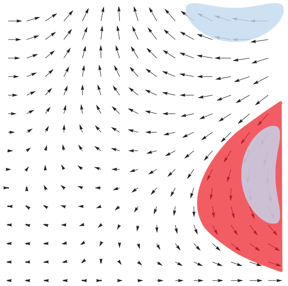
Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

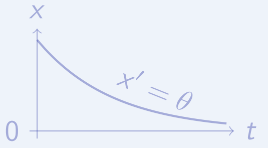
Differential Invariant

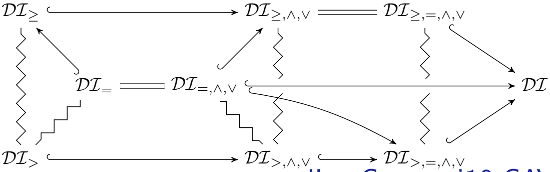


Differential Cut



Differential Ghost





Logic

Provability study

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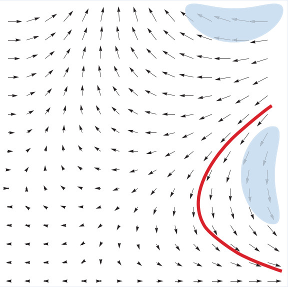
Character-istic PDE

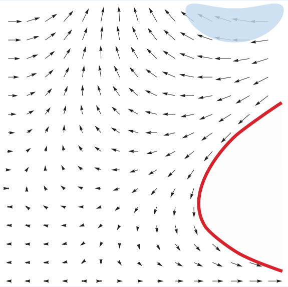
JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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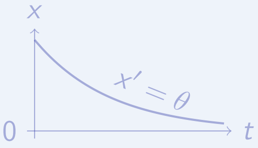

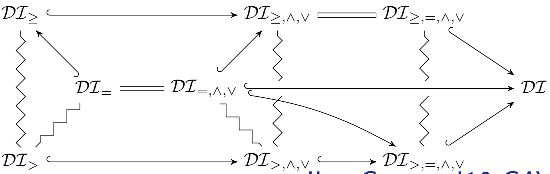

Logic

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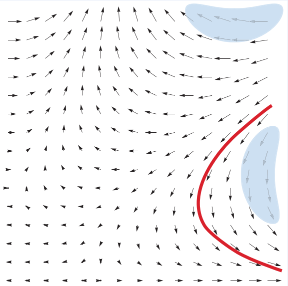
Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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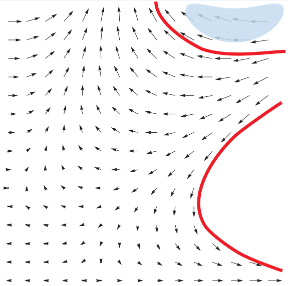
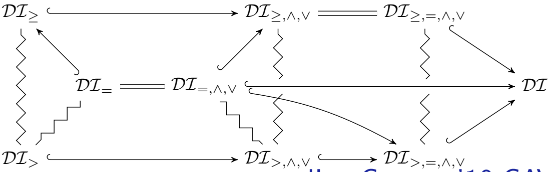


Differential Ghost



Logic

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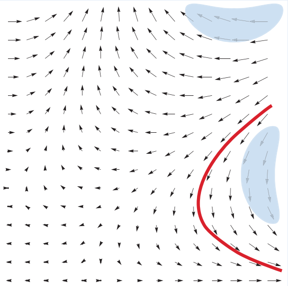
Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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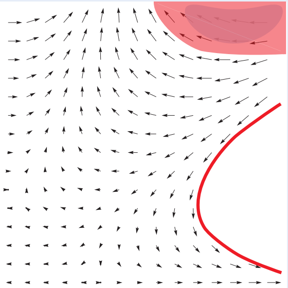
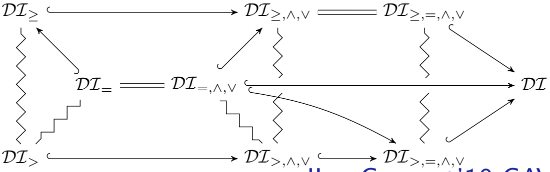


Differential Ghost



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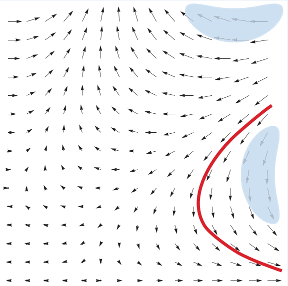
JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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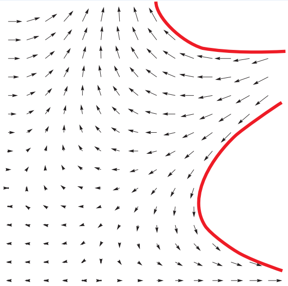

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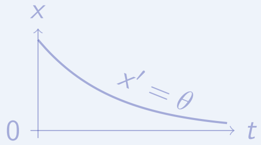

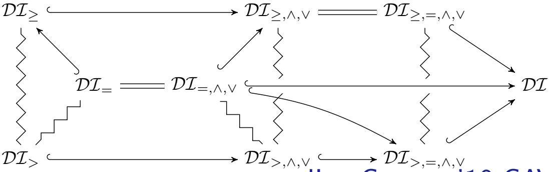

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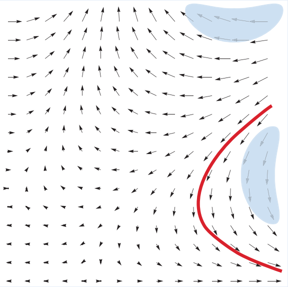
Differential Invariants for Differential Equations

Differential Invariant



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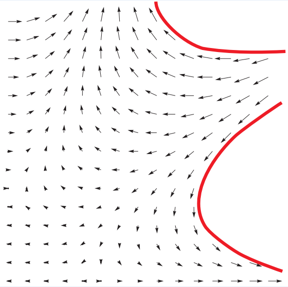

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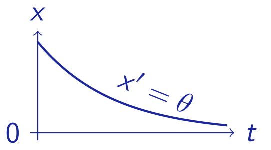

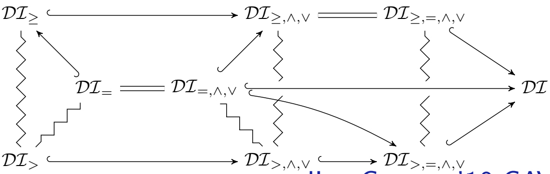

Logic

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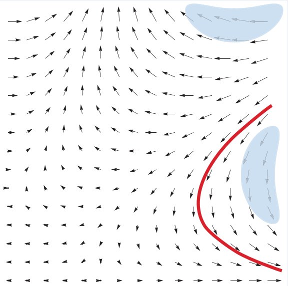
Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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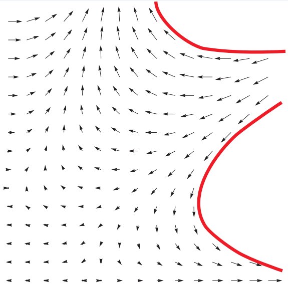

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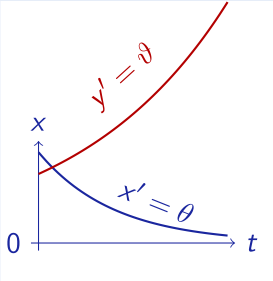
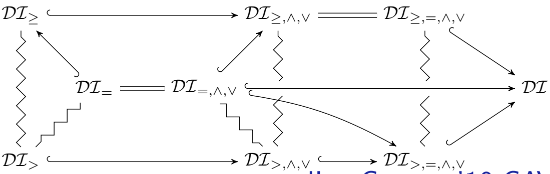




Logic

Provability study

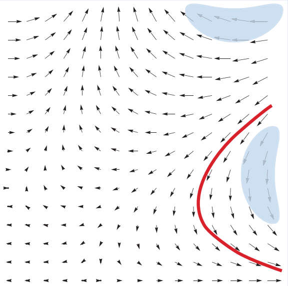
Math

Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Invariants for Differential Equations

Differential Invariant



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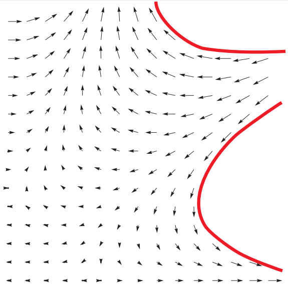

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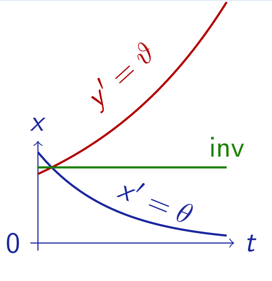

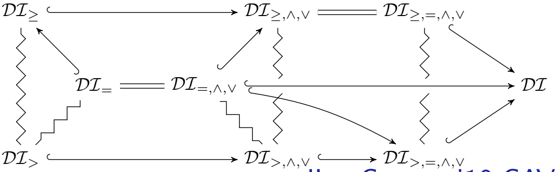

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Character-istic PDE

JLogComput'10,CAV'08,FMSD'09,LMCS'12,ITP'12

Differential Radical Invariants

Theorem (Differential radical invariant characterization)

$$\frac{h = 0 \to \bigwedge_{i=0}^{N-1} \mathcal{L}_p^{(i)}(h) = 0}{h = 0 \to [x' = p]h = 0}$$

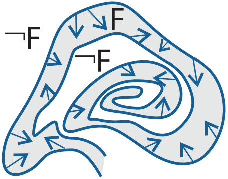
characterizes all algebraic invariants, where $N = ord \sqrt[\mathcal{L}]{(h)}$, i.e.

$$\mathcal{L}_p^{(N)}(h) = \sum_{i=0}^{N-1} g_i \mathcal{L}_p^{(i)}(h)$$

Corollary (Algebraic Invariants Decidable)

Invariance decidable for real algebraic $h = 0$.

TACAS'14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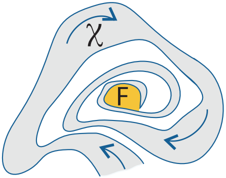
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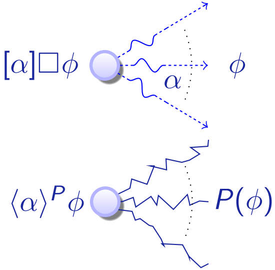
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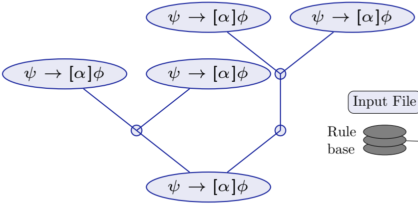
$\langle\alpha\rangle^P\phi$
$P(\phi)$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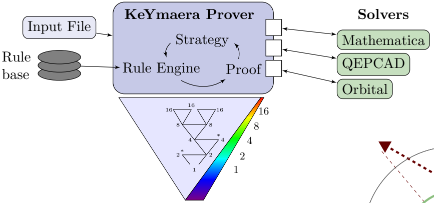
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Rule Engine
Proof
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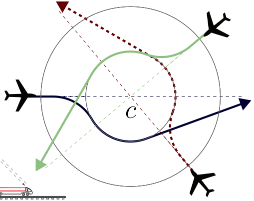
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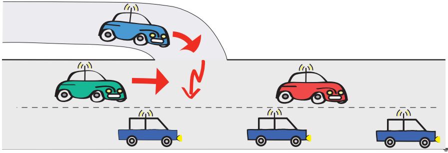

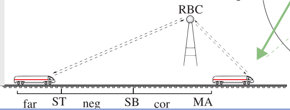
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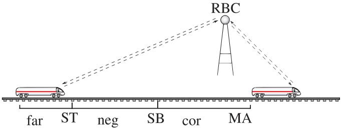
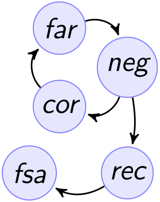
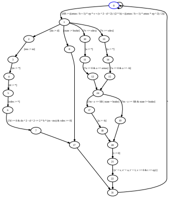
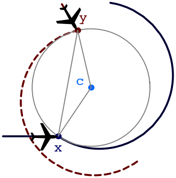
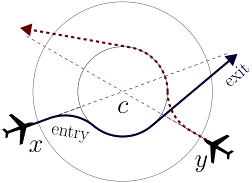
Successful CPS Proofs

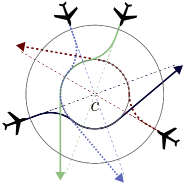


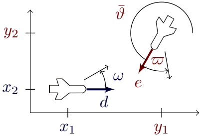


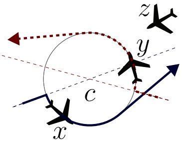


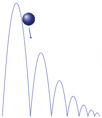


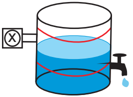






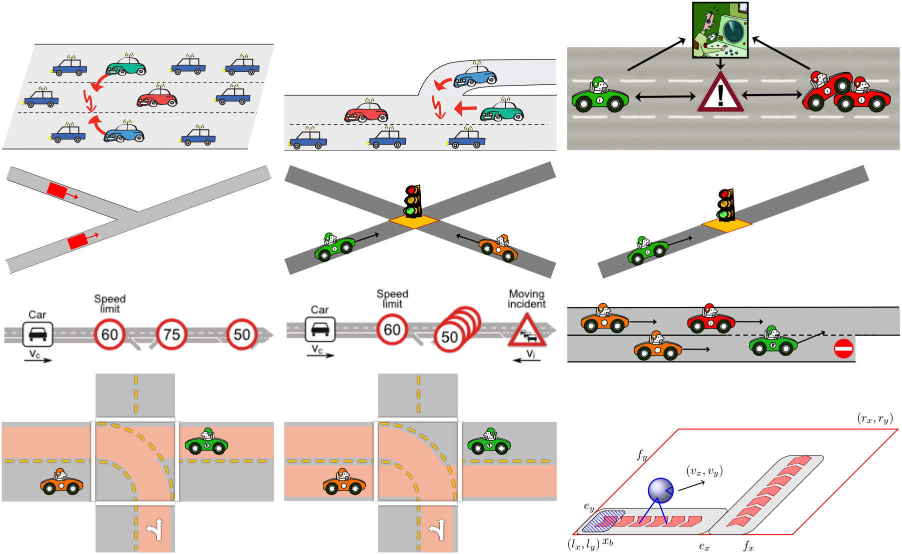






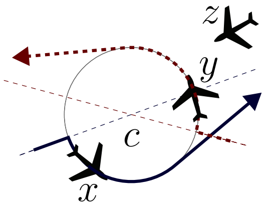
ICFEM'09,CAV'08,FM'09,HSCC'11

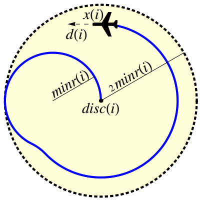
Successful CPS Proofs

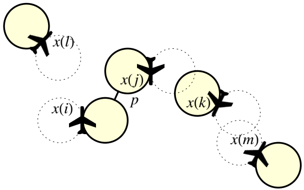


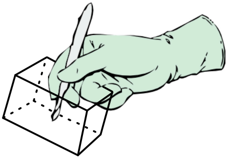
FM'11,LMCS'12,ICCPS'12,ITSC'11,ITSC'13,IJCAR'12

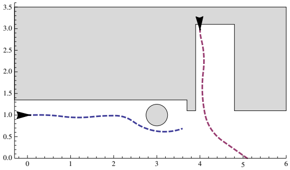
Successful CPS Proofs

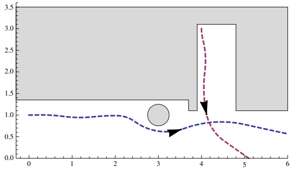


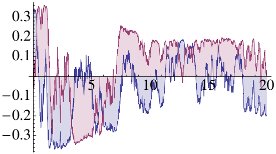


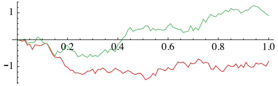


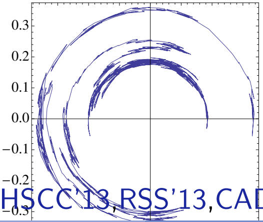




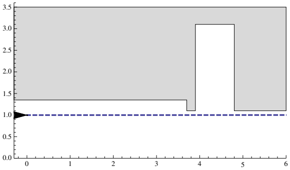


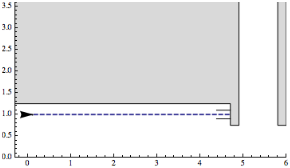


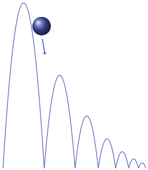


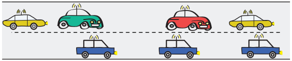


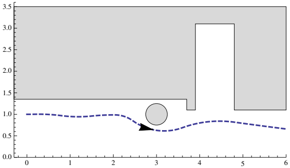
HSCC'11,HSCC'13,HSCC'13,RSS'13,CADE'12

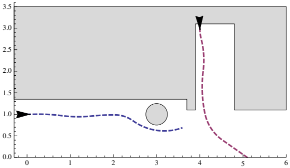


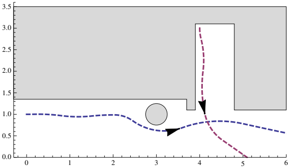








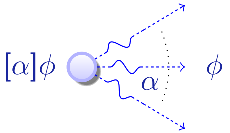


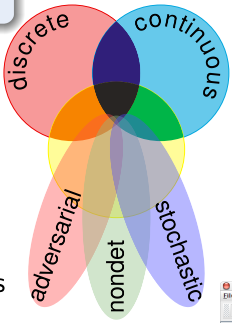


15-424/624 *Foundations of Cyber-Physical Systems* students

differential dynamic logic

$d\mathcal{L} = DL + HP$





- Multi-dynamical systems
- Combine simple dynamics
- Tame complexity
- Logic & proofs for CPS
- Theory of CPS
- Applications

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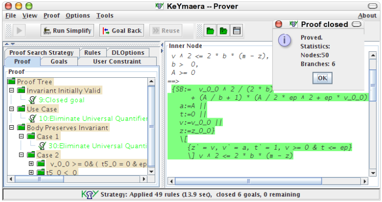

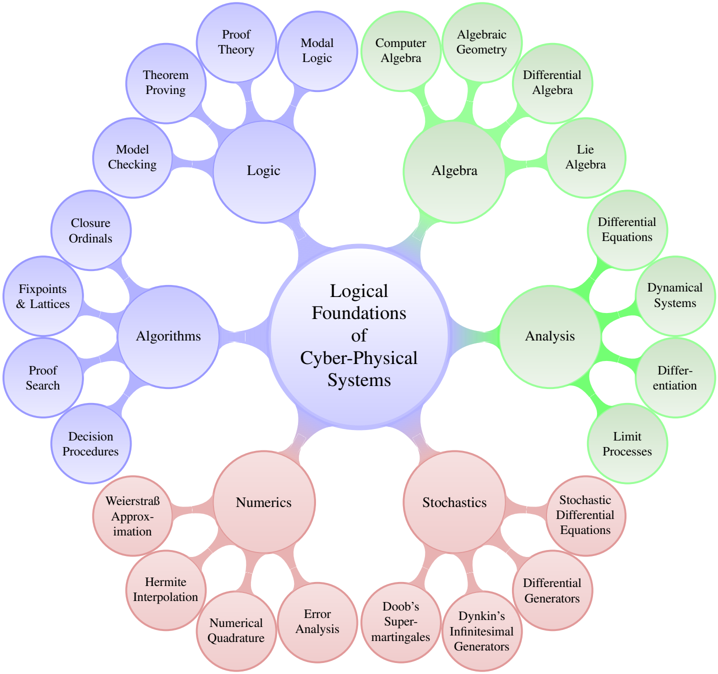
Logical Foundations of Cyber-Physical Systems
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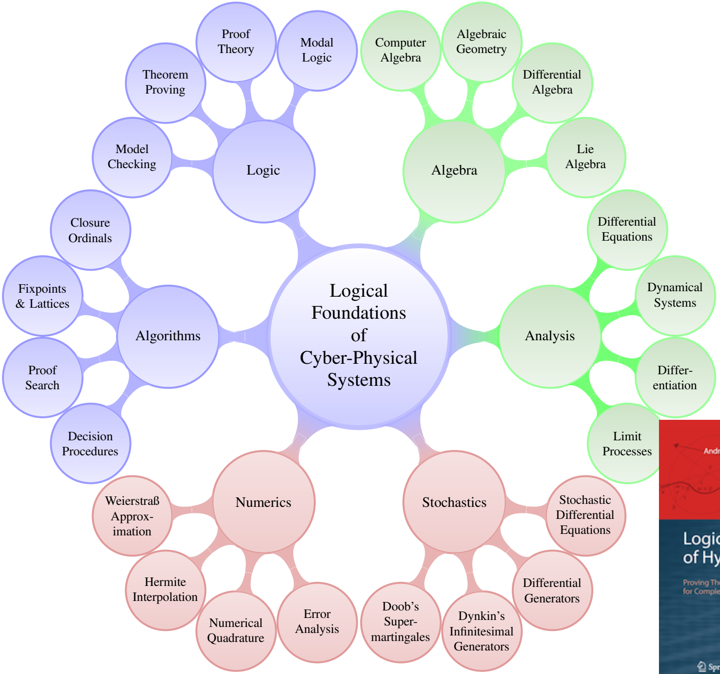
Proof Theory
Modal Logic
Theorem Proving
Model Checking
Algebra
Computer Algebra
Algebraic Geometry
Differential Algebra
Lie Algebra
Analysis
Differential Equations
Dynamical Systems
Differ-entiation
Limit Processes
Algorithms
Closure Ordinals
Fixpoints & Lattices
Proof Search
Decision Procedures
Numerics
Weierstraß Approx-imation
Hermite Interpolation
Numerical Quadrature
Error Analysis
Stochastics
Stochastic Differential Equations
Differential Generators
Doob's Super-martingales
Dynkin's Infinitesimal Generators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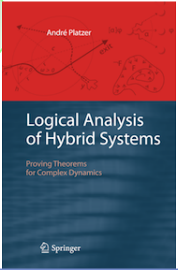
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