Modeling Cyber-Physical Systems Using Fractional-Order Differential Equations

Bill Goodwine

***Abstract*— Fractional-order differential equations are differential equations containing fractional-order derivative terms. Fractional-order derivatives are derivatives of non-integer order such as the "1/2 derivative" or "3/2 derivative." Fractional-order calculus dates back to nearly the beginning of calculus and has found application in a variety of scientific and engineering fields. This position paper presents two cases where fractional-order dynamics are present in cyber-physical systems, arising from the network structure in the first case and discrete dynamics in the second case, and argues that a deeper investigation of such modeling for CPS is warranted.**

I. Introduction

Fractional-order derivatives generalize the definition of integer-order derivatives which are pervasive in engineering analysis. There is not a unique way to accomplish this generalization with various approaches possessing relative advantages and corresponding disadvantages depending upon the application. This position paper will briefly summarize a numerical approach to the generalization and then show that, for two systems with features common to CPS, the responses are clearly fractional-order in nature. Because the features of the problems that give rise to the fractional-order dynamics are common in CPS, fractional-order modeling provides a possibly novel and valuable modeling tool for CPS.

Fractional calculus and fractional-order differential equations date back to near the foundations of calculus, and they has been used in engineering for several decades. Books include [1], [2] and some review articles are [3], [4]. The subject has also been studied to a certain degree in robotic and controls. Examples include [5] (walking robots), [6], [7] (flexible manipulators), [8] (time delays) and fractional-order PID control [7], [9].

It is, of course, natural to ask, given a function, $f(t)$ with a first derivative, $f^{(1)}(t)$ and second derivative, $f^{(2)}(t)$, whether there are operators "in between" the integer order derivatives such as the "one-half derivative," which generalizes the notion of an integer-order derivative. In this position paper, we consider only a numerical approach, but note that analytical formulations exist as well. To that end, if we consider the first and second derivatives of a function to be defined as

$$\frac{df}{dt}(t) = \lim_{\Delta t\to 0} \frac{f(t)-f(t-\Delta t)}{\Delta t}$$
$$\frac{d^2 f}{dt^2}(t) = \lim_{\Delta t\to 0} \frac{f(t)-2f(t-\Delta t)+f(t-2\Delta t)}{(\Delta t)^2}$$

or in general for an integer n

$$\frac{d^n f}{dt^n}(t) = \lim_{\Delta t\to 0} \frac{\sum_{0\leq m\leq n}(-1)^m \binom{n}{m} f\left(t+(n-m)\,\Delta t\right)}{(\Delta t)^n},$$

where the usual binomial coefficient is given by

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

Because the gamma function generalizes the factorial function to non-integers, we can replace the factorial functions in the binomial coefficient with gamma functions:

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma\left(\alpha-m+1\right)}.$$

Using this we arrive at the *Grünwald - Letnikov derivative*:

$$\frac{d^\alpha f}{dt^\alpha}(t) = \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f\left(t+(\alpha-j)\,\Delta t\right),$$

where α can have any real value and which can be numerically implemented for initial value problems with zero initial conditions and $\Delta t \ll 1$ by

$$\frac{d^\alpha f}{dt^\alpha}(t) \approx \frac{1}{(\Delta t)^\alpha} \sum_{j=0}^{[t/\Delta t]} (-1)^j \binom{\alpha}{j} f\left(t-j\Delta t\right). \quad (1)$$

For example, for

$$\frac{d^\alpha x}{dt^\alpha}(t) + 2x(t) = 1 \quad (2)$$

using Equation 1, Equation 2 is approximated by

$$\frac{1}{(\Delta t)^\alpha} \sum_{j=0}^{m} (-1)^j \binom{\alpha}{j} x\left((m-j)\Delta t\right) + 2x(m\Delta t) = 1.$$

Solving for $x(m\Delta t)$ gives

$$x(m\Delta t) \approx \frac{1-\frac{1}{(\Delta t)^\alpha}\sum_{j=1}^{m}(-1)^j \binom{\alpha}{j} x\left((m-j)\Delta t\right)}{2+\frac{1}{(\Delta t)^\alpha}}. \quad (3)$$

Solutions for various $\alpha \in [0.25, 2.0]$ and zero initial conditions are illustrated in Figure 1. The code (octave/Matlab) numerically computing these solutions is virtually trivial:

Bill Goodwine is with the Department of Aerospace & Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA, `bill@controls.ame.nd.edu`

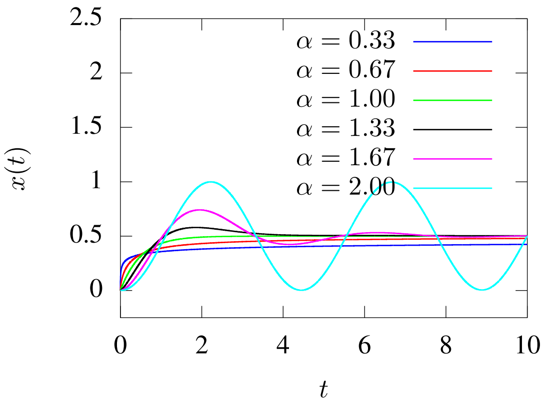

Fig. 1. Solution to Equation 2 using Equation 3. Note $\alpha = 1$ and $\alpha = 2$ give expected responses.

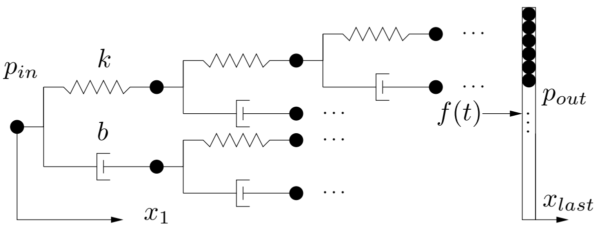

Fig. 2. Structure of vehicle formation.

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for alpha = [1/3 2/3 1 4/3 5/3 2]
  x = 0;
  coefs = 0;
  coefs(1) = -bincoeff(alpha,1);
  for i = 2:length(t)
    sum = dot(fliplr(x),coefs);
    x(i) = (1 - sum/(dt^alpha))/...
                (2 + 1/dt^alpha);
    coefs(i) = (-1)^i*bincoeff(alpha,i);
  end
end
```

II. Fractional-Order Dynamics in CPS

Next we present two examples with characteristics common to CPS transportation systems.

Fractional-Order Due to Network Structure: Consider a potential-driven formation control problem, schematically illustrated in Figure 2. Each vehicle in a lower generation is related to two in the subsequent generation where the control force between the lower vehicle and one of the next ones is represented by a linear spring and the relationship to the other vehicle is represented by a damper.

Under some reasonable simplifying assumptions, the transfer function relating the input to output is given by a repeated fraction [10]. Specifically, let

$$G(s) = \frac{X_1(s) - X_{last}(s)}{F(s)},$$

x_1 denote the position of the vehicle in the first generation and x_{last} denote the position of the vehicles in the last generation. If we let $G_1(s) = 1/k$ and $G_2(s) = 1/bs$, then the transfer function from the first to last generation is given

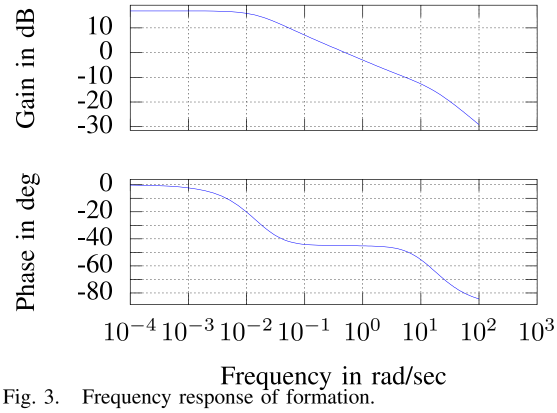

Fig. 3. Frequency response of formation.

by the repeated fraction

$$G(s) = \frac{1}{\frac{1}{G_1(s)+\frac{1}{\frac{1}{G_1(s)+\cdots}+\frac{1}{G_2(s)+\cdots}}} + \frac{1}{G_2(s)+\frac{1}{\frac{1}{G_1(s)+\cdots}+\frac{1}{G_s(s)+\cdots}}}}. \tag{4}$$

This transfer function has a denominator with order 2^N where N is the number of generations in the system. A Bode plot for a system with 6 generations is illustrated in Figure 3. The important feature of Figure 3 is that there is a large frequency range over which the slope of the magnitude plot is -10 db/decade and the phase is $-45°$, which suggests modeling the system by a transfer function containing a term of the form $\sqrt{s}$, or equivalently, a derivative of order $1/2$.

Note that in the limit of an infinite number of generations, the transfer function in Equation 4 may be written as [10]

$$G_\infty(s) = \frac{1}{\frac{1}{G_1(s)+G_\infty(s)} + \frac{1}{G_2(s)+G_\infty(s)}} = \sqrt{G_1(s)G_2(s)} = \sqrt{\frac{1}{kbs}}.$$

Hence, in the limit, we have a fractional-order relationship

$$\frac{X_1(s) - X_{last}(s)}{F(s)} = \left(\frac{1}{\sqrt{kb}}\right)\frac{1}{\sqrt{s}}.$$

If we take as the input to the network the position of the first vehicle, then we have

$$m_{last}s^2 X_{last}(s) = (X_1(s) - X_{last}(s))\sqrt{kbs},$$

the fractional-order transfer function

$$\frac{X_{last}(s)}{X_1(s)} = \frac{\sqrt{kbs}}{m_{last}s^2 + \sqrt{kbs}},$$

or in the time domain by

$$m\frac{d^2 x_{last}}{dt^2}(t) + \sqrt{kb}\frac{d^{\frac{1}{2}} x_{last}}{dt^{\frac{1}{2}}}(t) = \sqrt{kb}\frac{d^{\frac{1}{2}} x_1}{dt^{\frac{1}{2}}}(t).$$

Using the Grünwald - Letnikov definition for the fractional derivatives and solving for $x_{last}(t)$ gives the following

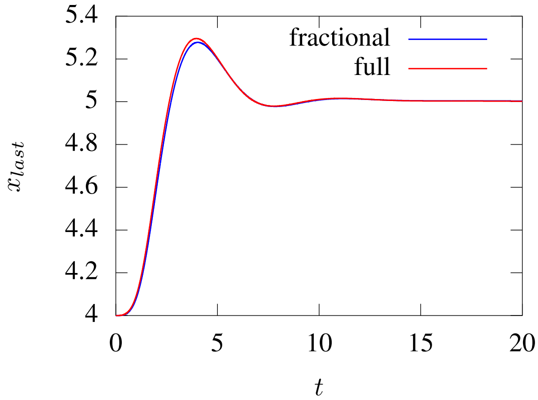

Fig. 4. Comparison of fractional-order solution with full solution for $k = b = 1$ and four generations.

numerical approximation at $t = n\Delta t$:

$$x_{last}(n\Delta t) \approx \left(\frac{1}{\frac{m}{(\Delta t)^2} + \frac{\sqrt{kb}}{\sqrt{\Delta t}}} \right) \times \left[\frac{m}{(\Delta t)^2} (2x((n-1)\Delta t) - x((n-2)\Delta t)) - \sum_{j=1}^{n} (-1)^j \binom{\frac{1}{2}}{j} x_{last}((n-j)\Delta t) + \frac{\sqrt{kb}}{\sqrt{\Delta t}} \sum_{j=0}^{n} (-1)^j \binom{\frac{1}{2}}{j} x_1((n-j)\Delta t) \right].$$

Figure 4 illustrates the response of a system with four generations versus the fractional-order response. Clearly the match is excellent.

Fractional-Order Due to Discrete Interactions: This example is motivated by the award-winning paper [11]. Consider a group of planar agents moving with constant velocity where each agent changes its heading at a rate proportional to the average heading of its neighbors. Two agents are neighbors only if they are within a specified distance of each other. Thus as the system of agents evolves, unless and until the agents converge to a common heading, for a specified agent, which of the other agents are neighbors will vary in time in a switching manner.

In the following simulation, we consider 25 agents, a radius defining neighbors as 0.1 and a proportional heading gain of 8. The domain has periodic boundary conditions so that if an agent travels out of the left boundary, it re-enters at the right, etc. One agent has a fixed heading of $\theta = 1$. All agents have random initial conditions in location and heading except the one agent with a fixed heading. If all agents had a random heading, we would expect the average heading would be near zero and each agent's heading would eventually converge to near zero. We can consider the case at hand, with one agent with a fixed heading of one, as the unit step response of the system.

The average heading of the agents averaged over 100 simulations is illustrated by the blue curve in Figure 5. The green curve is an exponential step response (it looks like a straight line due to the very large time constant). The red line

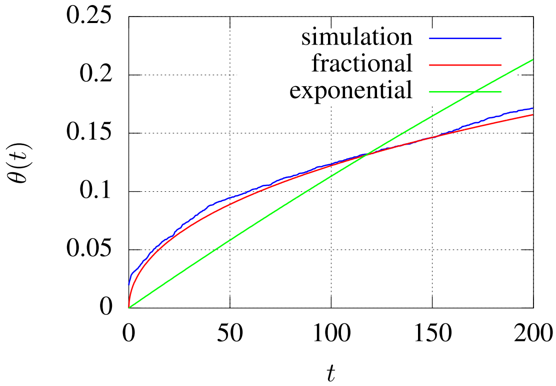

Fig. 5. Average heading of 25 agents (100 simulations).

which almost exactly matches the average heading response of the system is the solution to

$$\frac{d^{\frac{1}{2}}\theta}{dt^{\frac{1}{2}}}(t) + 0.012x(t) = 0.012,$$

indicating fractional order dynamics of order $1/2$.

III. Conclusions

This position paper presented two relatively simple examples of systems with characteristics common for cyber-physical transportation systems (and other CPS systems). Further investigation is warranted into the efficacy of this type of modeling for cyber-physical systems that have proved vexatious for engineering analysis and design.

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