Data-Driven Controller Synthesis via Finite Abstractions

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Continuous-Time Control Systems (ct-CS)

Consider a perturbed ct-CS $\mathbf{\Sigma}$ represented via $\mathbf{\Sigma} = (X, U, f, \mathbf{\Lambda})$, where:

- $X \subseteq \mathbb{R}^n$ is the state set and $U \subseteq \mathbb{R}^m$ is the input set;
- $f : X \times U \rightarrow X$ is the vector field;
- $\mathbf{\Lambda} = [-\bar{\lambda}, \bar{\lambda}] \subseteq \mathbb{R}^n$ is the disturbance set;

The state evolution of $\mathbf{\Sigma}$ is described by

$$\dot{x}(t) = f(x(t), \nu(t)) + \lambda(t),$$

with $\lambda(t) \in \mathbf{\Lambda}$.

$\xi_{x,u,\lambda}(\tau)$ denotes the state of $\mathbf{\Sigma}$ reached at time τ from initial state x under constant input $u \in U$ and disturbance $\lambda : [0, \tau] \rightarrow \mathbf{\Lambda}$.

Over-Approximating Reachable Sets

Given a ct-CS $\mathbf{\Sigma}$, and $\widehat{X} := [X]_{\eta_x}$, $\widehat{U} := [U]_{\eta_u}$ as symbolic state and input sets of $\mathbf{\Sigma}$, a function $\chi : \mathbb{R}^n_{\geq 0} \times \widehat{X} \times \widehat{U} \rightarrow \mathbb{R}^n_{\geq 0}$ satisfying

$$|\xi_{x',\hat{u},\lambda_1}(\tau) - \xi_{\hat{x},\hat{u},\lambda_2}(\tau)| \leq \chi(|x' - \hat{x}|, \hat{x}, \hat{u}),$$

$\forall \hat{x} \in \widehat{X}$, $\forall \hat{u} \in \widehat{U}$, $\forall x' \in \mathbf{\Phi}_{\eta_x}(\hat{x})$, where $\mathbf{\Phi}_{\eta_x}(\hat{x})$ is a ball centered at $\hat{x}$ with the radius $\eta_x \in \mathbb{R}^n_{\geq 0}$, $\forall \lambda_i : [0, \tau] \rightarrow \mathbf{\Lambda}$, $i \in \{1, 2\}$, is called a *growth bound* of $\mathbf{\Sigma}$.

Finite Abstractions

Given a ct-CS $\mathbf{\Sigma}$ and a growth bound χ, let $\mathbf{\Sigma}_\tau$ be the sampled-data version of $\mathbf{\Sigma}$ with the sampling time τ. Then $\widehat{\mathbf{\Sigma}} = (\widehat{X}, \widehat{U}, \hat{f})$ is a finite abstraction of $\mathbf{\Sigma}_\tau$, with the transition map $\hat{f} : \widehat{X} \times \widehat{U} \rightrightarrows \widehat{X}$ if:

- for any $\hat{x}, \hat{x}' \in \widehat{X}$ and $\hat{u} \in \widehat{U}$, $(\xi_{\hat{x},\hat{u},\lambda}(\tau) \oplus [-p', p']) \cap \mathbf{\Phi}_{\eta_x}(\hat{x}') \neq \emptyset \implies \hat{x}' \in \hat{f}(\hat{x}, \hat{u})$ where $p' = \chi(\eta_x, \hat{x}, \hat{u})$ and $\lambda : [0, \tau] \rightarrow \mathbf{\Lambda}$ is a disturbance signal.

Theorem 1: Feedback Refinement Relation

Consider a ct-CS $\mathbf{\Sigma}$ and its sampled-data version $\mathbf{\Sigma}_\tau$. Let $\widehat{\mathbf{\Sigma}} = (\widehat{X}, \widehat{U}, \hat{f})$ be a finite abstraction of $\mathbf{\Sigma}_\tau$. Then, $\mathbf{\Sigma}_\tau \preccurlyeq_{\mathcal{E}} \widehat{\mathbf{\Sigma}}$, i.e., the relation $\mathcal{E}$ defined as $(x, \hat{x}) \in \mathcal{E}$ if $x \in \mathbf{\Phi}_{\eta_x}(\hat{x})$ is a feedback refinement relation from $\mathbf{\Sigma}_\tau$ to $\widehat{\mathbf{\Sigma}}$.

We collect data from $\mathbf{\Sigma}$'s trajectories in one sampling time, which are collected in $\mathcal{D}_N := \{(x_l, u_l, x_l') \mid x_l' = \xi_{x_l,u_l,\lambda_l}(\tau),$ for some $\lambda_l : [0, \tau] \rightarrow \mathbf{\Lambda}$, $x_l \in X$, $u_l \in U$, $l = 1, 2, \ldots, N\}$.

Problem Statement

Consider a ct-CS $\mathbf{\Sigma}$ with an unknown vector field f. Develop a data-driven approach based on the set of data $\mathcal{D}_N$ for constructing a finite abstraction $\widehat{\mathbf{\Sigma}}$, such that $\mathbf{\Sigma}_\tau \preccurlyeq_{\mathcal{E}} \widehat{\mathbf{\Sigma}}$ with a set membership relation $\mathcal{E}$.

Data-Driven Construction of Finite Abstractions

Reachable set is over-approximated via proposed growth bound

$$\chi_\vartheta(\bar{s}, \hat{x}, \hat{u}) := \vartheta_1(\hat{x}, \hat{u})(\bar{s} + \bar{\lambda}\,\tau),$$

for any $\bar{s} \in \mathbb{R}^n_{\geq 0}$, $\hat{x} \in \widehat{X}, \hat{u} \in \widehat{U}$, where $\vartheta_1 \in \mathbb{R}^{n \times n}_{\geq 0}$, and $\vartheta \in \mathbb{R}^{n^2}_{\geq 0}$ is a column vector by stacking those of ϑ_1.

Computation of Growth Bound

The growth bound computation is formulated as a robust convex program (RCP):

$$\begin{cases} \min_\vartheta \quad \mathbf{1}^\top \vartheta \\ \text{s.t.} \quad \vartheta \in [0, \bar{\vartheta}], \forall x_1, x_2 \in \mathbf{\Phi}_{\eta_x}(\hat{x}),\ \forall \lambda_1, \lambda_2 : [0, \tau] \rightarrow \mathbf{\Lambda}, \\ \qquad |\xi(x_1, \hat{u}, \lambda_1) - \xi(x_2, \hat{u}, \lambda_2)| - \vartheta_1(\hat{x}, \hat{u})(|x_1 - x_2| + \bar{\lambda}\,\tau) \\ \qquad \leq \mathbf{0}, \end{cases}$$

where $\mathbf{1} \in \mathbb{R}^{n^2}$ and $\bar{\vartheta} \in \mathbb{R}^{n^2}_{\geq 0}$ is a sufficiently large vector component-wise.

Data-Driven Computation of Growth Bound

Since maps f is unknown, we use data set $\mathcal{D}_N$ and propose a scenario convex program (SCP) corresponding to the original RCP:

$$\text{SCP:} \begin{cases} \min_\vartheta \quad \mathbf{1}^\top \vartheta \\ \text{s.t.} \quad \vartheta \in [0, \bar{\vartheta}],\ \forall\, l, \bar{l} \in \{1, \ldots, N\}, \\ \qquad |x_l' - x_{\bar{l}}'| - \vartheta_1(\hat{x}, \hat{u})(|x_l - x_{\bar{l}}| + \bar{\lambda}\,\tau) + \varrho \leq \mathbf{0}, \end{cases}$$

where $\varrho \in \mathbb{R}^n_{\geq 0}$ is a bias term given in Theorem **2**.

Data-Driven Finite Abstraction

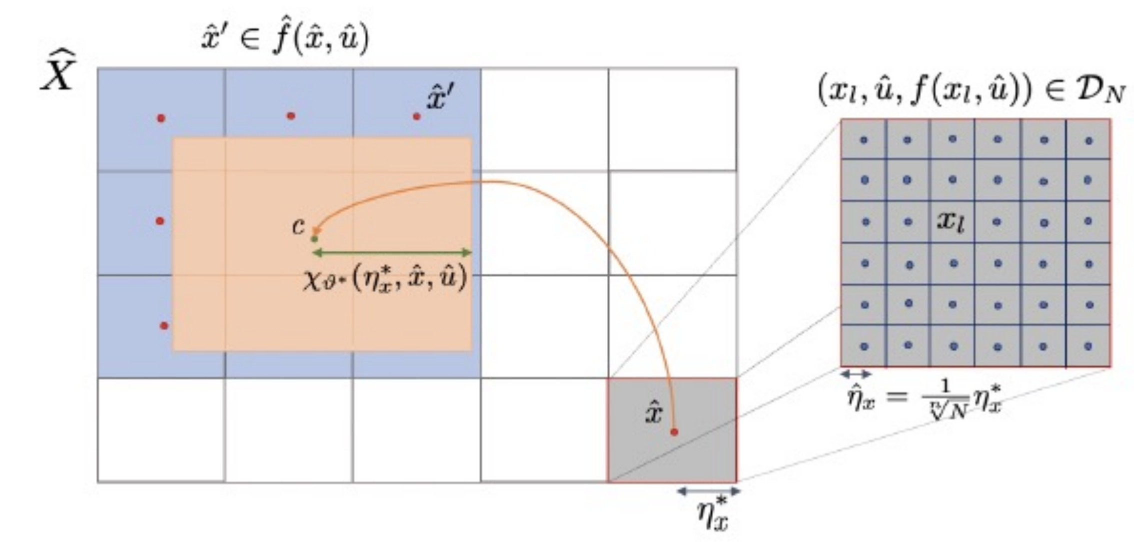


Figure: A 2-dimensional depiction of a finite abstraction, constructed using Algorithm 1.

Main Result

Theorem 2: Proposed growth bound χ_ϑ is sound

Consider a ct-CS $\mathbf{\Sigma}$ with a sampling time τ. For any $\hat{x} \in [X]_{\eta_x}$ and $\hat{u} \in [U]_{\eta_u}$, suppose $[\mathbf{\Phi}_{\eta_x}(\hat{x})]_{\hat{\eta}_x}$ is a finite partition of $\mathbf{\Phi}_{\eta_x}(\hat{x})$ where $\hat{\eta}_x := \frac{1}{\sqrt[n]{N}}\eta_x$. Then, the solution of SCP using the data from $[\mathbf{\Phi}_{\eta_x}(\hat{x})]_{\hat{\eta}_x}$ provides a growth bound χ_ϑ corresponding to $(\hat{x}, \hat{u})$ where

$$\varrho := 4(\mathcal{L}_x(\hat{u})\hat{\eta}_x + \mathcal{L}_\Lambda(\hat{u})\bar{\lambda}),$$

$\mathcal{L}_x(\hat{u})$ and $\mathcal{L}_\Lambda(\hat{u})$ are some Lipschitz constants.

Algorithm

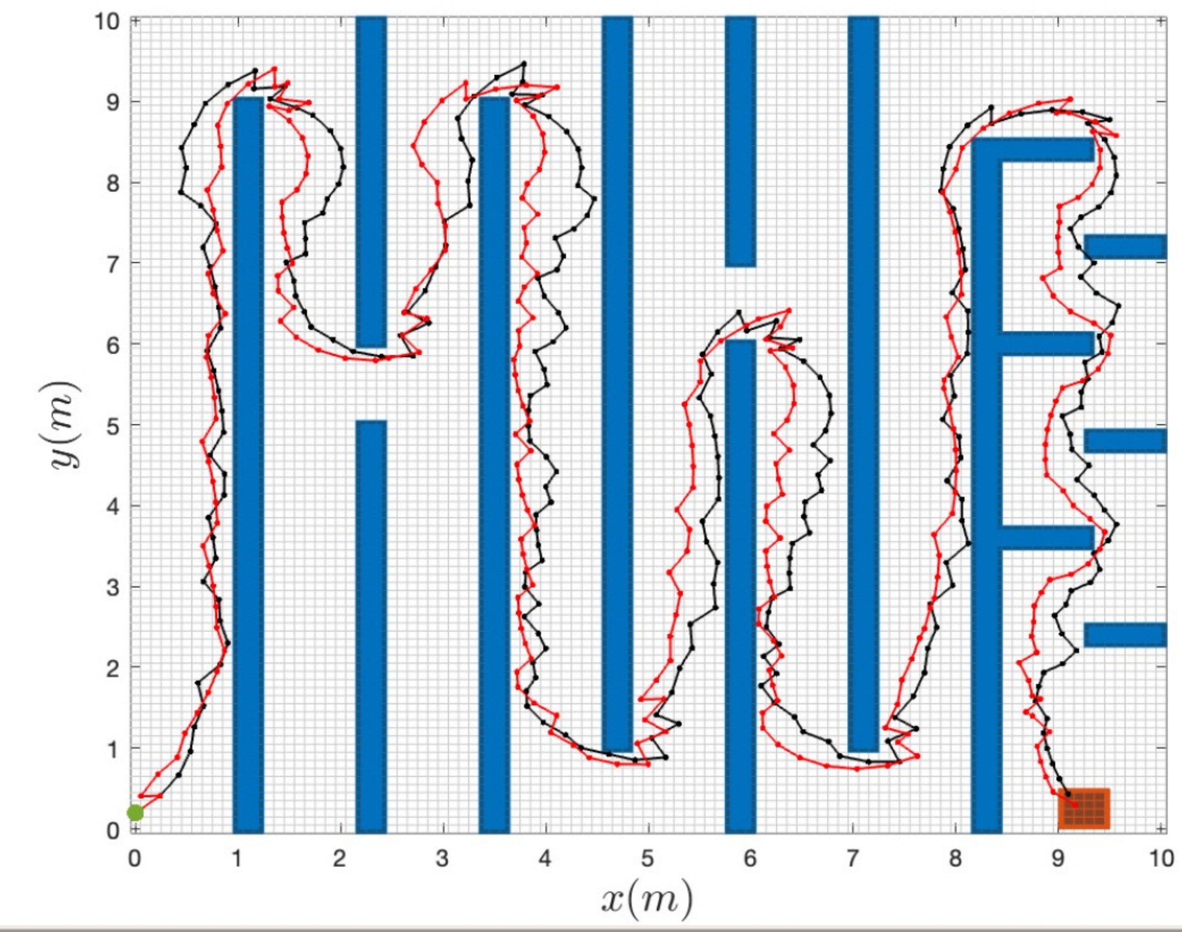
- Input: $X, U, \bar{\lambda}$, η_u and η_x
- Construct $\widehat{X} = [X]_{\eta_x}$ and $\widehat{U} = [U]_{\eta_u}$
- For each $(\hat{x}, \hat{u}) \in \widehat{X} \times \widehat{U}$ do:
  - Initiate $\hat{f}(\hat{x}, \hat{u}) = \emptyset$, $\rho = \mathbf{0} \in \mathbb{R}^n$ and $c = \xi_{\hat{x},\hat{u},\lambda}(\tau)$ for some $\lambda : [0, \tau] \rightarrow \mathbf{\Lambda}$
  - Compute $\varrho \in \mathbb{R}^n_{\geq 0}$ as in Theorem **2**
  - As outlined in Theorem **2**, generate $[\mathbf{\Phi}_{\eta_x}(\hat{x})]_{\hat{\eta}_x}$ and select N sampled data points $(x_l, \hat{u}, x_l')$ from it.
  - Obtain the optimal value $\vartheta^*(\hat{x}, \hat{u})$ of the SCP
  - Update: $\rho = \chi_{\vartheta^*}(\eta_x, \hat{x}, \hat{u})$
  - $\hat{f}(\hat{x}, \hat{u}) = \{\hat{x}' \in \widehat{X} \mid \mathbf{\Phi}_{\eta_x^*}(\hat{x}') \cap \mathbf{\Phi}_\rho(c) \neq \emptyset\} \cup \hat{f}(\hat{x}, \hat{u})$
- Output: Data-driven finite Abstraction $\widehat{\mathbf{\Sigma}} = (\widehat{X}, \widehat{U}, \hat{f})$

Case Study

Vehicle model, assumed to be unknown, is as:

$$\dot{x}(t) = \begin{bmatrix} u_1 \cos(q + x_3)/\cos(q) \\ u_1 \sin(q + x_3)/\cos(q) \\ u_1 \tan(u_2) \end{bmatrix} + \lambda(t),$$

where $q := \arctan(\tan(u_2)/2)$, $\lambda : [0, \tau] \rightarrow \mathbf{\Lambda} = [-\bar{\lambda}, \bar{\lambda}] \subset \mathbb{R}^3$ with $\bar{\lambda} = [0.15; 0.15; 0.015]$, $x(t) \in [0, 10]^2 \times [-\pi - 0.3798, \pi + 0.3798]$ and $u(t) \in [-1, 1]^2$.



Education and outreach

- Partnering with the Engineering GoldShirt and ASPIRE Summer Bridge Programs at CU Boulder to recruit first-year, first generation, and underrepresented engineering students and engage them in the platform used in this project.
- Incorporating the performed research tasks into the course entitled "Foundations of Autonomous Systems" and offering a Massively Open Online Course (MOOC) for it.

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