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FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

Multilateral Trades in Interconnected Power Systems: Localized Externalities

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Motivation and Challenges

Key issues in power networks

- Physical coupling of power flows according to Kirchhoff's laws
  - Cannot effectively be directly controlled
- Power companies/transmission operators possess private cost functions and network topology information
- Structural changes in the power industry

Vertically-integrated, Monopolistic	$\Rightarrow$	Decentralized, Competitive

These changes highlight the issue of asymmetric information.

Problem

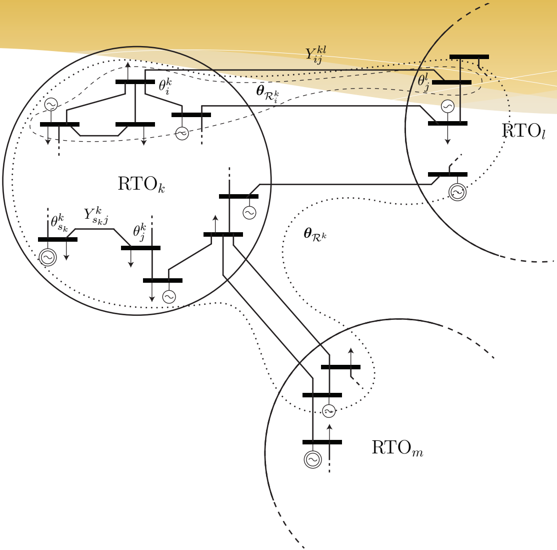
Consider a network of power system operators, termed Regional Transmission Operators (RTOs), which each own a collection on nodes, termed buses.

Each RTO contains generation buses and consumption buses, connected through a finite-capacity network.

Each RTO only has localized knowledge about the complete system.

- We wish to determine the optimal power trades between RTOs in order to minimize the sum of RTO's costs under the localized knowledge assumption.

The Model - Framework



The Model – Physical Assumptions

- Each bus $i \in \mathcal{N}^k$ has an associated (strictly increasing, convex) cost function, $c_i^k(p)$
  - If $p > 0$, $c_i^k(p)$ represents generation cost
  - If $p < 0$, $c_i^k(p)$ represents negative consumption benefit

- Flows are constrained

- RTOs are *non-strategic*

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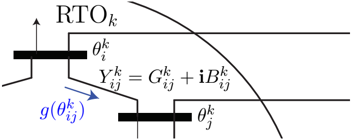
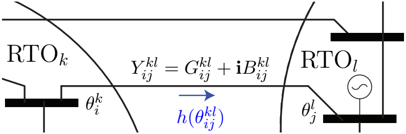
The Physical & Network Constraints

Intra-RTO$_k$ flow between buses i and j, $g(\theta_{ij}^k)$, and power trade between RTOs, $h(\theta_{ij}^{kl})$, defined as





$$g(\theta_{ij}^k) := B_{ij}^k(\theta_i^k - \theta_j^k) + \frac{1}{2}G_{ij}^k(\theta_i^k - \theta_j^k)^2 \qquad h(\theta_{ij}^{kl}) := B_{ij}^{kl}(\theta_i^k - \theta_j^l) + \frac{1}{2}G_{ij}^{kl}(\theta_i^k - \theta_j^l)^2$$

All (bidirectional) flows are constrained

$$g(\theta_{ij}^k) \le S_{ij}^k \qquad g(\theta_{ji}^k) \le S_{ji}^k \qquad h(\theta_{ij}^{kl}) \le S_{ij}^{kl} \qquad h(\theta_{ji}^{lk}) \le S_{ji}^{lk}$$

Power Injection

The total required real power injected, $P_i(\boldsymbol{\theta}_{\mathcal{R}_i^k})$, into a bus is equal to the power flowing *from* the bus *to* neighboring buses, given by

$$P_i(\boldsymbol{\theta}_{\mathcal{R}_i^k}) = \sum_{j \in \mathcal{R}_i^k \cap \mathcal{N}^k} g(\theta_{ij}^k) + \sum_{l \in \mathcal{S}^k} \sum_{j \in \mathcal{R}_i^k \cap \mathcal{N}^l} h(\theta_{ij}^{kl})$$

Thus, RTO$_k$'s (strictly convex) cost is

$$C_k(\boldsymbol{\theta}_{\mathcal{R}^k}) := \sum_{i \in \mathcal{N}^k} c_i^k \left(P_i \left(\boldsymbol{\theta}_{\mathcal{R}_i^k} \right) \right)$$

Centralized Information Problem

Assume there is some entity that has complete knowledge of the system. It would determine the optimal profile of voltage angles $\boldsymbol{\theta}^*$ via

$$\min_{\boldsymbol{\theta}} \; C(\boldsymbol{\theta}) = \sum_{k \in \mathcal{N}} C_k(\boldsymbol{\theta}_{\mathcal{R}^k}) \qquad (P_C)$$

subject to

$$\boldsymbol{\theta} \in \boldsymbol{\Theta} := \Big\{ \boldsymbol{\theta} \,\Big|\, \underline{\theta}_{ij}^k \leq \theta_{ij}^k \leq \overline{\theta}_{ij}^k,\, i \in \mathcal{N}^k;$$

$$g(\theta_{ij}^k) \leq S_{ij}^k,\, g(\theta_{ji}^k) \leq S_{ji}^k,\, (i,j) \in \mathcal{E}^k;$$

$$h(\theta_{ij}^{kl}) \leq S_{ij}^{kl},\, h(\theta_{ji}^{lk}) \leq S_{ji}^{lk},\, (i,j) \in \mathcal{E}^{kl}, l \in \mathcal{S}^k; k \in \mathcal{N} \Big\}$$

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The Informational Constraints

Each RTO_k knows

- **(I1)** The cost functions of the buses within their own region.
- **(I2)** The physical parameters of lines in, and immediately connected to, their region.
- **(I3)** The angle stability bounds of the buses in, and immediately neighboring, their region.

Decentralized Information Problem

The informational constraints impose additional restrictions on the problem.

$$\min_{\boldsymbol{\theta}} \; C(\boldsymbol{\theta}) = \sum_{k \in \mathcal{N}} C_k(\boldsymbol{\theta}_{\mathcal{R}^k}) \qquad (P_D)$$

$$\text{subject to} \quad \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

(I1), (I2), (I3) (*Localized info.*)

We aim to design an operational rule to find the optimal solution of Problem (P_D) which is the solution to Problem (P_C).

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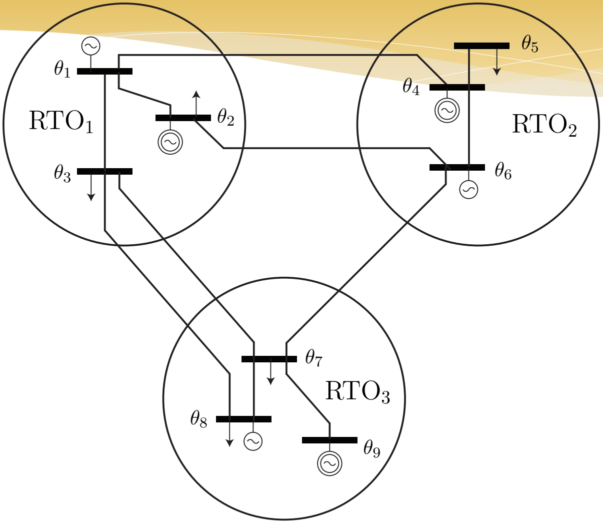
Solution Methodology - Local Public Goods

- We approach the problem using ideas from microeconomic theory
- Neighboring angles of bus i completely define how much bus i consumes/generates and its corresponding utility (localized externality effects)

Local public good $\Leftrightarrow$ Localized externality effects

- *Regional level generalization:* Each RTO proposes the voltage angles of all of its own buses and the angles of the buses immediately neighboring its region
  - Under our algorithm, agents eventually agree upon the socially optimal set of voltage angles.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RTO_1
RTO_2
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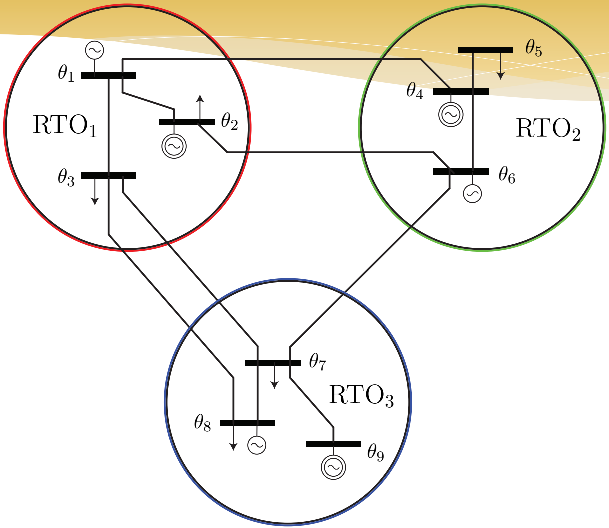

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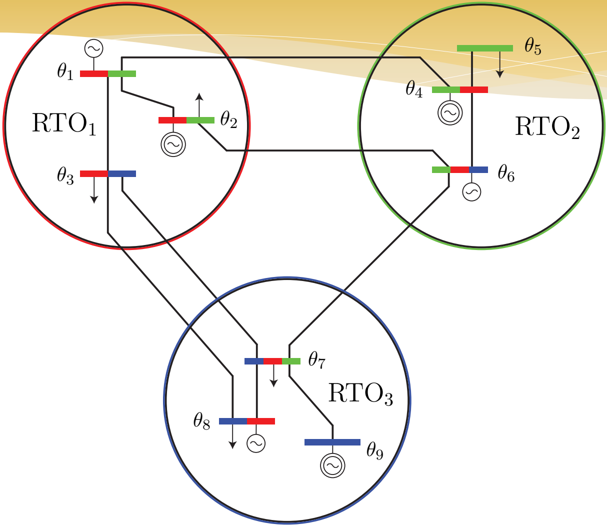

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RTO$_1$
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RTO$_3$
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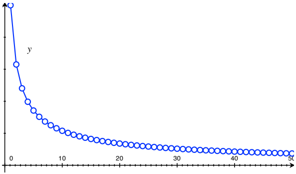
θ_9

Externality Algorithm

Step 0: Initialization

Set $n = 0$. Every RTO_k, $k \in \mathcal{N}$, agrees upon:

- An initial angle vector $\hat{\boldsymbol{\theta}}^{(0)} \in \boldsymbol{\Theta}$.
- A sequence of nonincreasing, bounded parameters, $\left\{\tau^{(n)}\right\}_{n=1}^{\infty}$



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Externality Algorithm

Step 1: Optimization and Broadcast

At the n^{th} iteration, each RTO_k, computes their *optimal* $(n+1)^{st}$ *proposal*:

$$\tilde{\boldsymbol{\theta}}_{\mathcal{R}^k}^{(n+1)} := \underset{\boldsymbol{\theta}_{\mathcal{R}^k} \in \hat{\boldsymbol{\Theta}}_{\mathcal{R}^k}}{\operatorname{argmin}} \left\{ C_k(\boldsymbol{\theta}_{\mathcal{R}^k}) + \frac{\left|\left| \boldsymbol{\theta}_{\mathcal{R}^k} - \hat{\boldsymbol{\theta}}_{\mathcal{R}^k}^{(n)} \right|\right|_2^2}{\tau^{(n+1)}} \right\}$$

and broadcasts parts of their solution to their neighbors.

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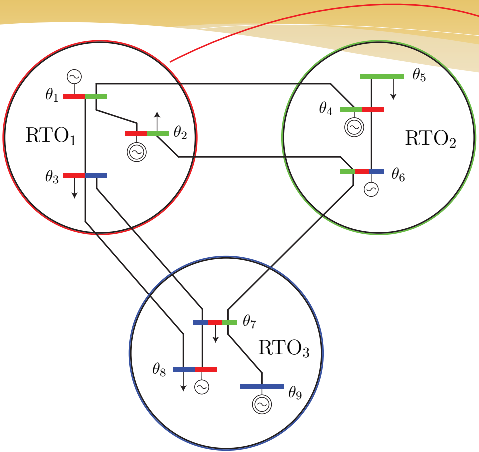
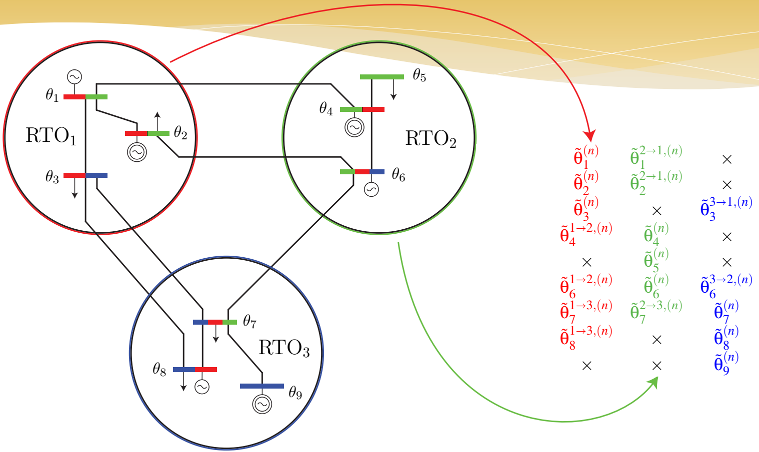


$\tilde{\theta}_1^{(n)}$	$\tilde{\theta}_1^{2\to1,(n)}$	$\times$
$\tilde{\theta}_2^{(n)}$	$\tilde{\theta}_2^{2\to1,(n)}$	$\times$
$\tilde{\theta}_3^{(n)}$	$\times$	$\tilde{\theta}_3^{3\to1,(n)}$
$\tilde{\theta}_4^{1\to2,(n)}$	$\tilde{\theta}_4^{(n)}$	$\times$
$\times$	$\tilde{\theta}_5^{(n)}$	$\times$
$\tilde{\theta}_6^{1\to2,(n)}$	$\tilde{\theta}_6^{(n)}$	$\tilde{\theta}_6^{3\to2,(n)}$
$\tilde{\theta}_7^{1\to3,(n)}$	$\tilde{\theta}_7^{2\to3,(n)}$	$\tilde{\theta}_7^{(n)}$
$\tilde{\theta}_8^{1\to3,(n)}$	$\times$	$\tilde{\theta}_8^{(n)}$
$\times$	$\times$	$\tilde{\theta}_9^{(n)}$



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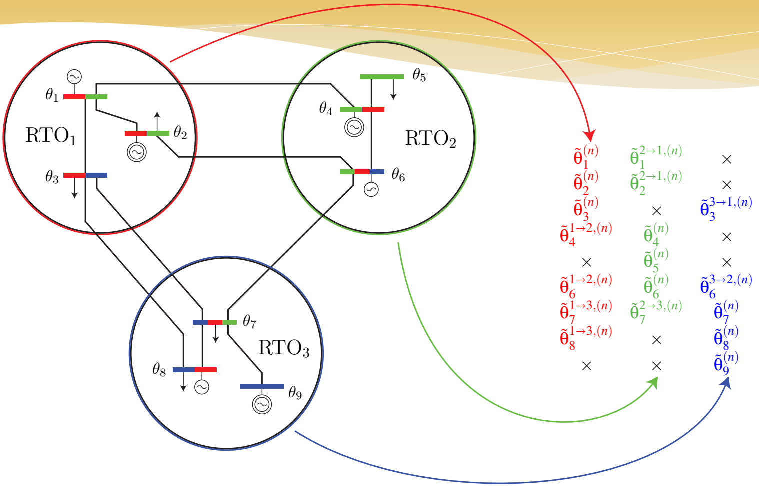
$\tilde{\theta}_8^{1\to3,(n)}$ $\times$ $\tilde{\theta}_8^{(n)}$
$\times$ $\times$ $\tilde{\theta}_9^{(n)}$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$\tilde{\theta}_7^{1\to3,(n)}$ $\tilde{\theta}_7^{2\to3,(n)}$ $\tilde{\theta}_7^{(n)}$
$\tilde{\theta}_8^{1\to3,(n)}$ $\times$ $\tilde{\theta}_8^{(n)}$
$\times$ $\times$ $\tilde{\theta}_9^{(n)}$

Externality Algorithm

Step 2: Averaging

Each RTO$_k$ computes averages of propositions

$\tilde{\theta}_1^{(n)}$	$\tilde{\theta}_1^{2\to1,(n)}$	$\times$
$\tilde{\theta}_2^{(n)}$	$\tilde{\theta}_2^{2\to1,(n)}$	$\times$
$\tilde{\theta}_3^{(n)}$	$\times$	$\tilde{\theta}_3^{3\to1,(n)}$
$\tilde{\theta}_4^{1\to2,(n)}$	$\tilde{\theta}_4^{(n)}$	$\times$
$\times$	$\tilde{\theta}_5^{(n)}$	$\times$
$\tilde{\theta}_6^{1\to2,(n)}$	$\tilde{\theta}_6^{(n)}$	$\tilde{\theta}_6^{3\to2,(n)}$
$\tilde{\theta}_7^{1\to3,(n)}$	$\tilde{\theta}_7^{2\to3,(n)}$	$\tilde{\theta}_7^{(n)}$
$\tilde{\theta}_8^{1\to3,(n)}$	$\times$	$\tilde{\theta}_8^{(n)}$
$\times$	$\times$	$\tilde{\theta}_9^{(n)}$

RTO$_1$:

$$\hat{\theta}_1^{(n)} = \frac{\tilde{\theta}_1^{(n)} + \tilde{\theta}_1^{2\to1,(n)}}{2}$$
$$\hat{\theta}_2^{(n)} = \frac{\tilde{\theta}_2^{(n)} + \tilde{\theta}_2^{2\to1,(n)}}{2}$$
$$\hat{\theta}_3^{(n)} = \frac{\tilde{\theta}_3^{(n)} + \tilde{\theta}_3^{3\to1,(n)}}{2}$$

RTO$_2$:

$$\hat{\theta}_4^{(n)} = \frac{\tilde{\theta}_4^{1\to2,(n)} + \tilde{\theta}_4^{(n)}}{2}$$
$$\hat{\theta}_5^{(n)} = \tilde{\theta}_5^{(n)}$$
$$\hat{\theta}_6^{(n)} = \frac{\tilde{\theta}_6^{1\to2,(n)} + \tilde{\theta}_6^{(n)} + \tilde{\theta}_6^{2\to3,(n)}}{3}$$

RTO$_3$:

$$\hat{\theta}_7^{(n)} = \frac{\tilde{\theta}_7^{1\to3,(n)} + \tilde{\theta}_7^{2\to3,(n)} + \tilde{\theta}_7^{(n)}}{3}$$
$$\hat{\theta}_8^{(n)} = \frac{\tilde{\theta}_8^{1\to3,(n)} + \tilde{\theta}_8^{(n)}}{2}$$
$$\hat{\theta}_9^{(n)} = \tilde{\theta}_9^{(n)}$$

Increment the iteration by setting $n = n + 1$ and return to Step 1.

Convergence is achieved asymptotically *but* we also need to ensure optimality.

To do this, each RTO computes a *weighted average*:

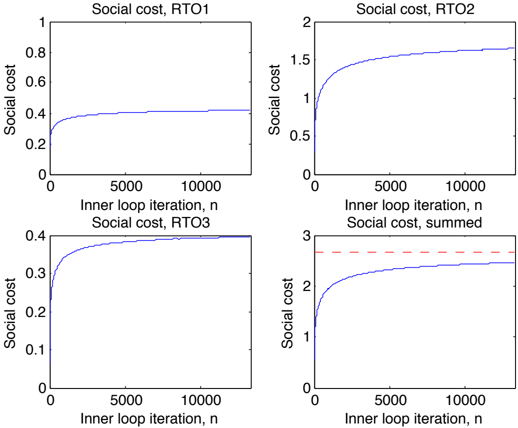
$$\boldsymbol{\omega}_{\mathcal{R}^k}^{(n+1)} := \frac{\sum_{m=1}^{n+1} \tau^{(m)} \tilde{\boldsymbol{\theta}}_{\mathcal{R}^k}^{(m)}}{\sum_{m=1}^{n+1} \tau^{(m)}}$$

$$\mathbf{W}^{(n)} = \begin{bmatrix} \omega_1^{(n)} & \omega_1^{2\to1,(n)} & 0 \\ \omega_2^{(n)} & \omega_2^{2\to1,(n)} & 0 \\ \omega_3^{(n)} & 0 & \omega_3^{3\to1,(n)} \\ \omega_4^{1\to2,(n)} & \omega_4^{(n)} & 0 \\ 0 & \omega_5^{(n)} & 0 \\ \omega_6^{1\to2,(n)} & \omega_6^{(n)} & \omega_6^{3\to2,(n)} \\ \omega_7^{1\to3,(n)} & \omega_7^{2\to3,(n)} & \omega_7^{(n)} \\ \omega_8^{1\to3,(n)} & 0 & \omega_8^{(n)} \\ 0 & 0 & \omega_9^{(n)} \end{bmatrix} \longrightarrow \mathbf{W}^{*} = \begin{bmatrix} \omega_1^{*} & \omega_1^{2\to1,*} & 0 \\ \omega_2^{*} & \omega_2^{2\to1,*} & 0 \\ \omega_3^{*} & 0 & \omega_3^{3\to1,*} \\ \omega_4^{1\to2,*} & \omega_4^{*} & 0 \\ 0 & \omega_5^{*} & 0 \\ \omega_6^{1\to2,*} & \omega_6^{*} & \omega_6^{3\to2,*} \\ \omega_7^{1\to3,*} & \omega_7^{2\to3,*} & \omega_7^{*} \\ \omega_8^{1\to3,*} & 0 & \omega_8^{*} \\ 0 & 0 & \omega_9^{*} \end{bmatrix}$$

The vector $\mathbf{w}^* = (w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*, w_7^*, w_8^*.w_9^*)$ is the solution to the centralized information problem, (P_C).

Numerical Example

Three RTO system, each with 2 buses.



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Conclusion & Future Work

- Designed an operational rule, when under asymmetric information, the decentralized optimization obtains the same solution as the centralized optimization
- *Take-away message:* choice of decision variable determines the amount of information exchange required to obtain the optimal solution

Future work:

- These algorithms are slow; future work involves increasing efficiency
- Currently working on the extension to strategic entities

Thank you! Questions?

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Proof of Strict Convexity

Lemma: Each RTO's aggregated cost, $C_k(\boldsymbol{\theta}_{\mathcal{R}^k})$, $k \in \mathcal{N}$, is strictly convex in $\boldsymbol{\theta}_{\mathcal{R}^k}$.

Proof.

For $i \in \mathcal{N}^k$, define the matrix $\mathbf{A}_k^i \in \mathbb{R}^{|\mathcal{R}_i^k| \times |\mathcal{R}^k|}$

$$\mathbf{A}_k^i = \begin{bmatrix} \text{——} & \mathbf{e}_i^\top - \mathbf{e}_{[\mathcal{R}_i^k]_1}^\top & \text{——} \\ \text{——} & \mathbf{e}_i^\top - \mathbf{e}_{[\mathcal{R}_i^k]_2}^\top & \text{——} \\ & \vdots & \\ \text{——} & \mathbf{e}_i^\top - \mathbf{e}_{[\mathcal{R}_i^k]_{|\mathcal{R}_i^k|}}^\top & \text{——} \end{bmatrix}$$

where $\mathbf{e} \in \mathbb{R}^{|\mathcal{R}^k|-1}$ are the standard basis vectors.

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Proof.

Now, construct the matrix

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{A}_k^1 \\ \mathbf{A}_k^2 \\ \vdots \\ \mathbf{A}_k^{N_k+1} \end{bmatrix} \in \mathbb{R}^{(\sum_i |\mathcal{R}_i^k|) \times |\mathcal{R}^k|}$$

Remove the s_k^{th} column of $\mathbf{A}_k$ to form the matrix $\bar{\mathbf{A}}_k \in \mathbb{R}^{(\sum_i |\mathcal{R}_i^k|) \times (|\mathcal{R}^k| - 1)}$.
This is due to the fact that the s_k^{th} column corresponds to $\theta_{s_k}^k$, which takes on the reference value $\theta_{s_k}^k = 0$. Note that the resulting matrix is of full rank, $\text{rank}(\bar{\mathbf{A}}_k) = |\mathcal{R}^k| - 1$.
Define the composition $C_k = B_k \circ \bar{\mathbf{A}}_k$ and perform the change of variables

$$\boldsymbol{\phi}^k = \bar{\mathbf{A}}_k \boldsymbol{\theta}_{\mathcal{R}^k \setminus \{s_k\}}$$

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Proof.

We now show the strict convexity of $B_k(\boldsymbol{\phi}^k)$ in $\boldsymbol{\phi}^k$.

The vector of $\boldsymbol{\phi}^k$ represents the angle differences within and immediately neighboring RTO$_k$. Due to the functional form of the power flow equations, the variables $\boldsymbol{\phi}_{\mathcal{R}_i^k} = \phi_{[\mathcal{R}_i^k]_1}, \phi_{[\mathcal{R}_i^k]_2}, \ldots, \phi_{[\mathcal{R}_i^k]_{|\mathcal{R}_i^k|}}$ of the arguments of c_i^k appear as positively weighted sums of linear and (element-wise) squared terms (a strictly convex function). By assumption, each c_i^k is strictly increasing and convex. Since the composition of a strictly increasing, convex function and a strictly convex function is strictly convex [S. Boyd and L. Vandenberghe, 2004], the restricted maps below are strictly convex.

$$\boldsymbol{\phi}_{\mathcal{R}_i^k} \mapsto c_i^k\left(f_i^k\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right), i \in \mathcal{N}^k \backslash \mathcal{N}_b^k$$

$$\boldsymbol{\phi}_{\mathcal{R}_i^k} \mapsto c_i^k\left(f_i\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right), i \in \mathcal{N}_b^k$$

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Proof.

Define

$$d_i^k(\boldsymbol{\phi}_{\mathcal{R}^k}) := c_i^k\left(f_i^k\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right), i \in \mathcal{N}^k \backslash \mathcal{N}_b^k$$
$$d_i^k(\boldsymbol{\phi}_{\mathcal{R}^k}) := c_i^k\left(f_i\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right), i \in \mathcal{N}_b^k$$

By construction, each $d_i^k(\boldsymbol{\phi}_{\mathcal{R}^k})$, $i \in \mathcal{N}^k \backslash \mathcal{N}_b^k$, and $d_i(\boldsymbol{\phi}_{\mathcal{R}^k})$, $i \in \mathcal{N}_b^k$, is convex in $\boldsymbol{\phi}_{\mathcal{R}^k}$. Observe

$$\begin{aligned} B_k(\boldsymbol{\phi}_{\mathcal{R}^k}) &= \sum_{i \in \mathcal{N}^k \backslash \mathcal{N}_b^k} c_i^k\left(f_i^k\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right) + \sum_{i \in \mathcal{N}_b^k} c_i^k\left(f_i\left(\boldsymbol{\phi}_{\mathcal{R}_i^k}\right)\right) \\ &= \sum_{i \in \mathcal{N}^k} d_i^k(\boldsymbol{\phi}_{\mathcal{R}^k}) \end{aligned}$$

Proof.

Let $\mathbf{x} \neq \mathbf{y}$, $\lambda \in (0,1)$, then for all $i \in \mathcal{N}^k$

$$d_i^k(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda d_i^k(\mathbf{x}) + (1-\lambda) d_i^k(\mathbf{y}) \tag{1}$$

Since $\mathbf{x} \neq \mathbf{y}$, then $x_p \neq y_p$ for at least one p, so at least one of the $N_k + 1$ inequalities in (1) is strict. Since $B_k(\boldsymbol{\phi}_{\mathcal{R}^k}) = \sum_{i \in \mathcal{N}^k} d_i^k(\boldsymbol{\phi}_{\mathcal{R}^k})$ we have

$$B_k(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda B_k(\mathbf{x}) + (1-\lambda) B_k(\mathbf{y}) \tag{2}$$

This establishes the strict convexity of $B_k(\boldsymbol{\phi}_{\mathcal{R}^k})$ in $\boldsymbol{\phi}_{\mathcal{R}^k}$. Since $\bar{\mathbf{A}}$ is a linear transformation of full-rank, we equivalently have that $C_k(\boldsymbol{\theta}_{\mathcal{R}^k})$ is strictly convex in $\boldsymbol{\theta}_{\mathcal{R}^k}$.

$\square$

Constraints on Injections

Model allows for constraints on power injections $f_i^k(\boldsymbol{\theta}_{\mathcal{R}_i^k}) \in [\underline{M}_i^k, \bar{M}_i^k]$.

Can be approximated by the following convex inequalities

$$f_i^k(\boldsymbol{\theta}_{\mathcal{R}_i^k}) = \sum_{j \in \mathcal{R}_i^k} g(\theta_{ij}^k) \leq \bar{M}_i^k \quad \text{and} \quad \sum_{j \in \mathcal{R}_i^k} g(\theta_{ji}^k) \leq \underline{M}_i^k$$

Notice that

$$\left\{ \boldsymbol{\theta}_{\mathcal{R}_i^k} \,\middle|\, \sum_{j \in \mathcal{R}_i^k} g(\theta_{ji}^k) \leq \underline{M}_i^k \right\} \subseteq \left\{ \boldsymbol{\theta}_{\mathcal{R}_i^k} \,\middle|\, \sum_{j \in \mathcal{R}_i^k} g(\theta_{ij}^k) \geq \underline{M}_i^k \right\}$$

Power Consumption

Let $h_i^k(\boldsymbol{\theta}_{\mathcal{R}_i^k})$ be the power consumption at bus $i \in \mathcal{N}^k$ defined as

$$
\begin{aligned}
h_i^k(\boldsymbol{\theta}_{\mathcal{R}_i^k}) &= \sum_{j \in \mathcal{R}_i^k} [g(\theta_{ji}^k) - L(\theta_{ji}^k)] \\
&= \sum_{j \in \mathcal{R}_i^k} [B_{ji}^k(\theta_j^k - \theta_i^k) + \frac{1}{2} G_{ji}^k(\theta_j^k - \theta_i^k)^2 - G_{ji}^k(\theta_j^k - \theta_i^k)^2] \\
&= \sum_{j \in \mathcal{R}_i^k} [-B_{ij}^k(\theta_i^k - \theta_j^k) - \frac{1}{2} G_{ij}^k(\theta_i^k - \theta_j^k)^2] \\
&= -\sum_{j \in \mathcal{R}_i^k} g(\theta_{ij}^k) = -f_i^k(\boldsymbol{\theta}_{\mathcal{R}_i^k})
\end{aligned}
$$

...a concave function in $\boldsymbol{\theta}_{\mathcal{R}_i^k}$.

Memory of One

$$\boldsymbol{\omega}_{\mathcal{R}^k}^{(n+1)} := \frac{1}{\sigma^{(n+1)}} \sum_{m=1}^{n+1} \tau^{(m)} \tilde{\boldsymbol{\theta}}_{\mathcal{R}^k}^{(m)} = \frac{\tau^{(n+1)} \tilde{\boldsymbol{\theta}}_{\mathcal{R}^k}^{(n+1)} + \sigma^{(n)} \boldsymbol{\omega}_{\mathcal{R}^k}^{(n)}}{\sigma^{(n+1)}} \tag{3}$$