# Robust Convergence of Distributed Routing with Heterogeneous Population Dynamics

**Walid Krichene** **Alex Bayen**

Dept. of Electrical Engineering & Computer Sciences,
UC Berkeley, CA, USA

# Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results
2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)
3. Simulations



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results
2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)
3. Simulations



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Routing game

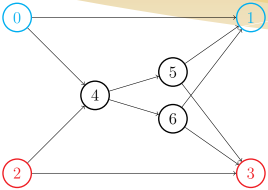



Figure: Example network

- Directed graph $(V, E)$
- Population $k$: paths $\mathcal{P}_k$
- Population distribution over paths $x_{\mathcal{P}_k} \in \Delta^{\mathcal{P}_k}$
- Loss on path $p$: $\ell_p(x)$



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

## Routing game

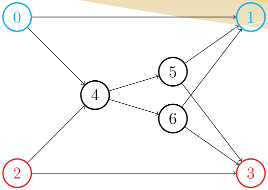



Figure: Example network

- Directed graph $(V, E)$
- Population $k$: paths $\mathcal{P}_k$
- Population distribution over paths $x_{\mathcal{P}_k} \in \Delta^{\mathcal{P}_k}$
- Loss on path $p$: $\ell_p(x)$



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Routing game

**Equilibrium**

$x^* = (x^*_{\mathcal{P}_1}, \ldots, x^*_{\mathcal{P}_k})$ is an equilibrium if $\forall k$,

$$\left\langle \ell_{\mathcal{P}_k}(x^*), x^*_{\mathcal{P}_k} \right\rangle \leq \left\langle \ell_{\mathcal{P}_k}(x^*), x_{\mathcal{P}_k} \right\rangle$$

Losses are minimal on the support of $x^*_{\mathcal{P}_k}$



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# The routing game

## One-shot routing game

- Well understood
- Useful for characterizing 'steady-state' behavior
  - Network performance (price of anarchy)
  - System optimal tolling
  - Other applications

Why study dynamics?

- How do players arrive at equilibrium?
- How fast?
- Stability?
- Robustness (noisy measurements)?



FORCES
FOUNDATIONS OF RESILIENT
CYBER-PHYSICAL SYSTEMS

## The routing game

### One-shot routing game

- Well understood
- Useful for characterizing 'steady-state' behavior
  - Network performance (price of anarchy)
  - System optimal tolling
  - Other applications

Why study dynamics?

- How do players arrive at equilibrium?
- How fast?
- Stability?
- Robustness (noisy measurements)?

# Applications

| | Transportation networks | Packet routing | Load balancing |
|---|---|---|---|
| Time scale | Day | minute/second | minute/second |
| Measurements | Route delays | Route delays | Job completion |
| Decision model | Distributed | Distributed | Can be centralized |



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Catastrophic failure: Mississippi river bridge collapse (2005)



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Catastrophic failure: Mississippi river bridge collapse (2005)

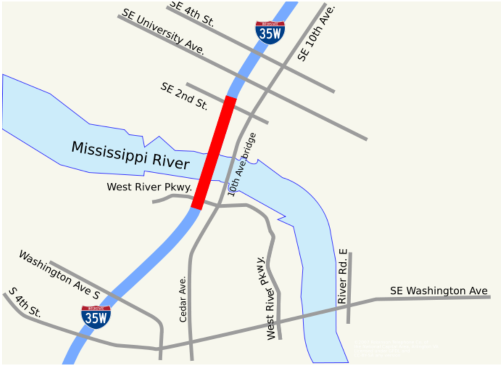


FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Catastrophic failure: Mississippi river bridge collapse (2005)






FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Incident response: closure of I15 after fire on bridge during construction.



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Adding a link to the network: construction of the Millau Viaduct (2004)



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Adding a link to the network: construction of the Millau Viaduct (2004)




FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rate

How fast does the system reconverge to equilibrium?

- Tolling: Electronic Road Pricing (ERP) in Singapore.




FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Existing results

Continuous time:

- General case of potential games, under a positive correlation condition [9]
- Special case of routing games, under replicator dynamics [5]

Discrete time:

- General class of no-regret dynamics, limited convergence result [2]

---

[9] William H Sandholm. Potential games with continuous player sets. *Journal of Economic Theory*, 97(1):81–108, 2001

[5] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004

[2] Avrim Blum, Eyal Even-Dar, and Katrina Ligett. Routing without regret: on convergence to nash equilibria of regret-minimizing algorithms in routing games. In *Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing*, PODC '06, pages 45–52, New York, NY, USA, 2006. ACM

# Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results
2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)
3. Simulations

# Routing game

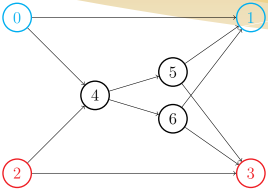



Figure: Example network

- Directed graph $(V, E)$
- Population $k$: paths $\mathcal{P}_k$
- Population distribution over paths $x_{\mathcal{P}_k} \in \Delta^{\mathcal{P}_k}$
- Loss on path $p$: $\ell_p(x)$



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Online learning model

## Online learning model

At iteration $t$

- Players of population $k$ choose routes. Distribution $x^{(t)}$.
- $\ell_{\mathcal{P}_k}(x^{(t)})$ is revealed to players of population $k$.
- Players update their distribution.

$$x_{\mathcal{P}_k}^{(t+1)} = u_k(x_{\mathcal{P}_k}^{(t)}, \text{history})$$

## Main problem

Define a class $\mathcal{C}$ of algorithms (update rules) such that

$$u_k \in \mathcal{C} \ \forall k \Rightarrow x^{(t)} \to \mathcal{N}$$

Extension: Losses are noisy $\hat{\ell}_{\mathcal{P}_k}(x^{(t)})$ with

$$\mathbb{E}[\hat{\ell}_{\mathcal{P}_k}(x^{(t)}) | x^{(t)}] = \ell_{\mathcal{P}_k}(x^{(t)})$$



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Online learning model

## Online learning model

At iteration $t$

- Players of population $k$ choose routes. Distribution $x^{(t)}$.
- $\ell_{\mathcal{P}_k}(x^{(t)})$ is revealed to players of population $k$.
- Players update their distribution.

$$x_{\mathcal{P}_k}^{(t+1)} = u_k(x_{\mathcal{P}_k}^{(t)}, \text{history})$$

## Main problem

Define a class $\mathcal{C}$ of algorithms (update rules) such that

$$u_k \in \mathcal{C} \ \forall k \Rightarrow x^{(t)} \to \mathcal{N}$$

Extension: Losses are noisy $\hat{\ell}_{\mathcal{P}_k}(x^{(t)})$ with

$$\mathbb{E}[\hat{\ell}_{\mathcal{P}_k}(x^{(t)}) | x^{(t)}] = \ell_{\mathcal{P}_k}(x^{(t)})$$



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convex potential

## Rosenthal potential

$$f(x) \text{ Convex}$$
$$\nabla_{x_{\mathcal{P}_k}} f(x) = \ell_{\mathcal{P}_k}(x)$$
$$\mathcal{N} = \arg\min_{x \in \Delta^{\mathcal{P}_1} \times \cdots \times \Delta^{\mathcal{P}_K}} f(x)$$

Optimality conditions:

$$\langle \ell(x^*), x - x^* \rangle \geq 0 \quad \forall x \qquad \Leftrightarrow \qquad \forall k,\ \forall x_{\mathcal{P}_k}, \langle \ell_{\mathcal{P}_k}(x^*_{\mathcal{P}_k}), x_{\mathcal{P}_k} - x^*_{\mathcal{P}_k} \rangle \geq 0$$

- Continuous time: $f$ used as a Lyapunov function.
- Discrete time: regret.



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Outline

# Regret

Instantaneous regret:

$$r^{(t)}(x) = \left\langle \ell(x^{(t)}), x^{(t)} - x \right\rangle$$

## Equilibrium

$$x^{(t)} \to \mathcal{N} \Leftrightarrow \limsup_{t} \sup_{x} r^{(t)}(x) \leq 0$$



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Regret

Average cumulative regret

$$R^{(t)}(x) = \frac{1}{t} \sum_{\tau \leq t} r^{(\tau)}(x)$$

## Equilibrium

$$\bar{x}^{(t)} = \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)} \to \mathcal{N} \Leftrightarrow \limsup_{t} \sup_{x} R^{(t)}(x) \leq 0$$

By convexity of $f$,

$$f\left(\frac{1}{t} \sum_{\tau \leq t} x^{(\tau)}\right) - f(x) \leq \frac{1}{t} \sum_{\tau \leq t} f(x^{(\tau)}) - f(x) \leq \frac{1}{t} \sum_{\tau \leq t} \left\langle \ell(x^{(t)}), x^{(t)} - x \right\rangle = R^{(t)}(x)$$



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Regret

- Regret first defined by Hannan (1957) in the context of repeated games [6]
- Large classes of algorithms have "no regret" guarantees, e.g. [3]
- However, only guarantees convergence of $\bar{x}^{(t)}$, not $x^{(t)}$
- Seek additional conditions to guarantee $x^{(t)} \to \mathcal{N}$.

**Observation**

If $f(x^{(t)})$ is eventually monotone, then $f(x^{(t)}) \to f^*$.

[6] James Hannan. Approximation to Bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–139, 1957

[3] Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games.* Cambridge University Press, 2006



FORCES
FOUNDATIONS OF RESILIENT
CYBER-PHYSICAL SYSTEMS

# Regret

- Regret first defined by Hannan (1957) in the context of repeated games [6]
- Large classes of algorithms have “no regret” guarantees, e.g. [3]
- However, only guarantees convergence of $\bar{x}^{(t)}$, not $x^{(t)}$
- Seek additional conditions to guarantee $x^{(t)} \to \mathcal{N}$.

## Observation

If $f(x^{(t)})$ is eventually monotone, then $f(x^{(t)}) \to f^*$.

---

[6] James Hannan. Approximation to Bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–139, 1957

[3] Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games.* Cambridge University Press, 2006



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results
2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)
3. Simulations

# Replicator dynamics

## Replicator equation

$$\forall p \in \mathcal{P}_k, \frac{dx_p^k}{dt} = x_p^k \left( \langle \ell_{\mathcal{P}_k}(x), x_{\mathcal{P}_k} \rangle - \ell_p^k(x) \right) \quad (1)$$

Also in evolutionary game theory, Weibull [10].

## Theorem: Fischer and Vöcking [5]

Every solution of the ODE (1) converges to the set of its stationary points.

Proof: $f$ is a Lyapunov function.

[10] Jörgen W Weibull. *Evolutionary game theory.* MIT press, 1997

[5] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

## Replicator dynamics

### Replicator equation

$$\forall p \in \mathcal{P}_k, \frac{dx_p^k}{dt} = x_p^k \left( \langle \ell_{\mathcal{P}_k}(x), x_{\mathcal{P}_k} \rangle - \ell_p^k(x) \right) \quad (1)$$

Also in evolutionary game theory, Weibull [10].

### Theorem: Fischer and Vöcking [5]

Every solution of the ODE (1) converges to the set of its stationary points.

Proof: $f$ is a Lyapunov function.

---

[10] Jörgen W Weibull. *Evolutionary game theory.* MIT press, 1997

[5] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Approximate REPlicator update

**Discretization of the continuous-time replicator dynamics**

$$x_p^{(t+1)} - x_p^{(t)} = \eta_t x_p^{(t)} \left( \left\langle \ell^k(x^{(t)}), x_{\mathcal{P}_k}^{(t)} \right\rangle - \ell^k(x^{(t)}) \right) + \eta_t U_p^{(t+1)}$$

$(U^{(t)})_{t \geq 1}$ perturbations that satisfy for all $T > 0$,

$$\lim_{\tau_1 \to \infty} \max_{\tau_2 : \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0$$

Michel Benaïm. Dynamics of stochastic approximation algorithms. In *Séminaire de probabilités XXXIII*, pages 1–68. Springer, 1999



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

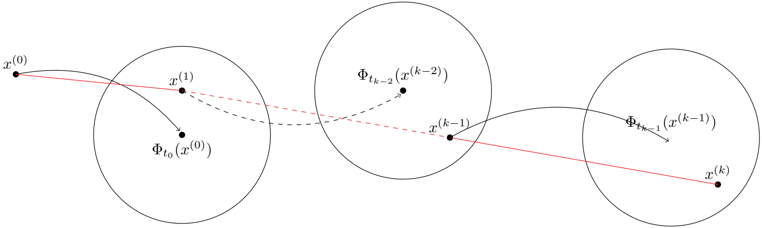
# Convergence to Nash equilibria

**Theorem Krichene et al. [7]**

Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x^{(t)} \to \mathcal{N}$$

- Affine interpolation of $x^{(t)}$ is an asymptotic pseudo trajectory.




- $f$ is a Lyapunov function for Nash equilibria in the continuous system.

However, No convergence rates.

---

[7] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 163–171. JMLR Workshop and Conference Proceedings, 2014

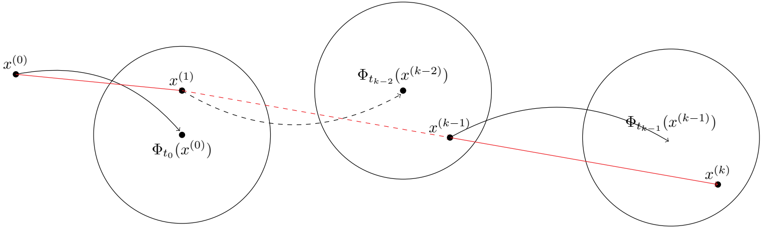
# Convergence to Nash equilibria

**Theorem Krichene et al. [7]**

Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x^{(t)} \to \mathcal{N}$$

- Affine interpolation of $x^{(t)}$ is an asymptotic pseudo trajectory.




- $f$ is a Lyapunov function for Nash equilibria in the continuous system.

However, No convergence rates.

---

[7] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 163–171. JMLR Workshop and Conference Proceedings, 2014



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results
2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)
3. Simulations

# Stochastic Mirror Descent

| | | |
|---|---|---|
| minimize | $f(x)$ | convex function |
| subject to | $x \in \mathcal{X} \subset \mathbb{R}^d$ | convex, compact set |

At iteration $t$

- have a stochastic subgradient $\hat{g}^{(t)}$
- $\hat{g}^{(t)}$ unbiased: $\mathbb{E}\left[\hat{g}^{(t)} | \mathcal{F}_t\right] = g^{(t)} \in \partial f(x^{(t)})$ ($\mathcal{F}_t$ natural filtration of $(x^{(t)})$)

**Algorithm 1** SMD Method with learning rates $(\eta_t)$

1: **for** $t \in \mathbb{N}$ **do**
2: $\quad \hat{g}^{(t)} \in \mathcal{F}_{t+1}$
3: $\quad x^{(t+1)} = \arg\min_{x \in \mathcal{X}} \left\langle \hat{g}^{(t)}, x \right\rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)})$
4: **end for**



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Stochastic Mirror Descent

| | | |
|---|---|---|
| minimize | $f(x)$ | convex function |
| subject to | $x \in \mathcal{X} \subset \mathbb{R}^d$ | convex, compact set |

At iteration $t$

- have a stochastic subgradient $\hat{g}^{(t)}$
- $\hat{g}^{(t)}$ unbiased: $\mathbb{E}\left[\hat{g}^{(t)} | \mathcal{F}_t\right] = g^{(t)} \in \partial f(x^{(t)})$ ($\mathcal{F}_t$ natural filtration of $(x^{(t)})$)

**Algorithm 2** SMD Method with learning rates $(\eta_t)$

1: **for** $t \in \mathbb{N}$ **do**
2: $\quad \hat{g}^{(t)} \in \mathcal{F}_{t+1}$
3: $\quad x^{(t+1)} = \arg\min_{x \in \mathcal{X}} \left\langle \hat{g}^{(t)}, x \right\rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)})$
4: **end for**



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Stochastic Mirror Descent

| | | |
|---|---|---|
| minimize | $f(x)$ | convex function |
| subject to | $x \in \mathcal{X} \subset \mathbb{R}^d$ | convex, compact set |

At iteration $t$

- have a stochastic subgradient $\hat{g}^{(t)}$
- $\hat{g}^{(t)}$ unbiased: $\mathbb{E}\left[\hat{g}^{(t)} | \mathcal{F}_t\right] = g^{(t)} \in \partial f(x^{(t)})$ ($\mathcal{F}_t$ natural filtration of $(x^{(t)})$)

**Algorithm 3** SMD Method with learning rates $(\eta_t)$

1: **for** $t \in \mathbb{N}$ **do**
2: $\quad \hat{g}^{(t)} \in \mathcal{F}_{t+1}$
3: $\quad x^{(t+1)} = \arg\min_{x \in \mathcal{X}} \left\langle \hat{g}^{(t)}, x \right\rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)})$
4: **end for**



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Mirror Descent

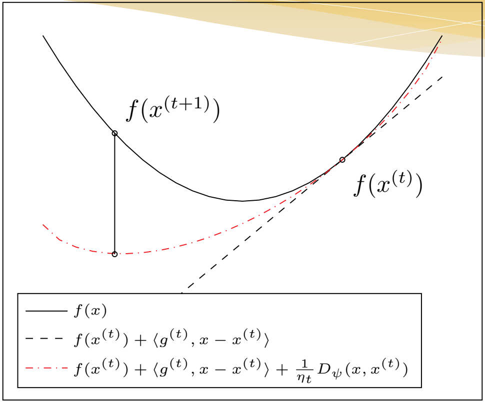



Figure: Mirror Descent iteration

# Bregman Divergence

## Bregman Divergence

Strongly convex function $\psi$

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$$

- $\psi(x) = \frac{1}{2}\|x\|_2^2$, $D_\psi(x, y) = \frac{1}{2}\|x - y\|_2^2$ (projected gradient)
- $\psi(x) = -H(x) = \sum_{i=1}^{d} x_i \ln x_i$, $D_\psi(x, y) = D_{KL}(x, y) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}$.

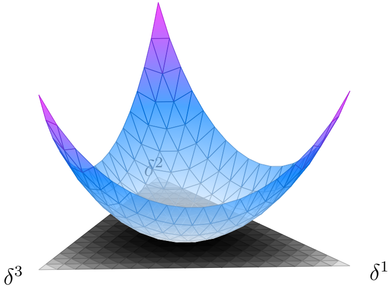



Figure: KL divergence



FORCES
FOUNDATIONS OF RESILIENT
CYBER-PHYSICAL SYSTEMS

# Bregman Divergence

## Bregman Divergence

Strongly convex function $\psi$

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x - y \rangle$$

- $\psi(x) = \frac{1}{2}\|x\|_2^2$, $D_\psi(x, y) = \frac{1}{2}\|x - y\|_2^2$ (projected gradient)
- $\psi(x) = -H(x) = \sum_{i=1}^{d} x_i \ln x_i$, $D_\psi(x, y) = D_{KL}(x, y) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}$.

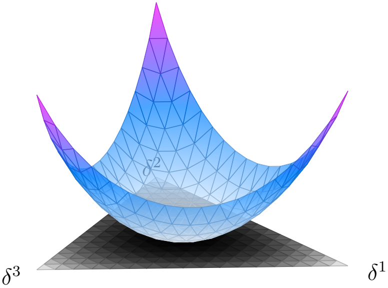



Figure: KL divergence

# Bregman Divergence

## Bregman Divergence

Strongly convex function $\psi$

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x - y \rangle$$

- $\psi(x) = \frac{1}{2}\|x\|_2^2$, $D_\psi(x, y) = \frac{1}{2}\|x - y\|_2^2$ (projected gradient)
- $\psi(x) = -H(x) = \sum_{i=1}^d x_i \ln x_i$, $D_\psi(x, y) = D_{KL}(x, y) = \sum_{i=1}^d x_i \ln \frac{x_i}{y_i}$.

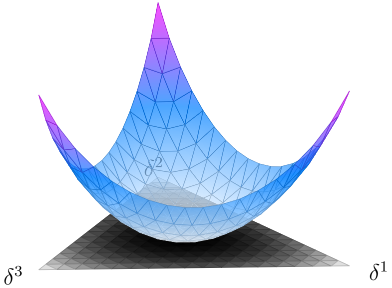



Figure: KL divergence



FORCES FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Convergence rates

| $f$ | $\eta_t$ | Convergence |
|---|---|---|
| Weakly convex | $\frac{c}{\sqrt{t}}$ | $\frac{1}{t}\sum_{\tau=1}^{t}\mathbb{E}\left[f(x^{(\tau)})\right] - f^\star = O(\frac{1}{\sqrt{t}})$ [8] |
| | $\frac{c}{t^\alpha},\ \alpha \in (0,1)$ | $\mathbb{E}\left[f(x^{(t)})\right] - f^\star = O\left(\frac{\log t}{t^{\min(\alpha,1-\alpha)}}\right)$ |
| Strongly convex | $\eta_t \to 0,\ \sum \eta_t = \infty$ | $\mathbb{E}\left[D_\psi(x^\star, x^{(t)})\right] = O\left(\eta_T + e^{-\sum_T^t \eta_\tau}\right)$ |
| | $\frac{\theta}{\ell_f t^\alpha},\ \alpha \in (0,1)$ | $\mathbb{E}\left[D_\psi(x^\star, x^{(t)})\right] = O(t^{-\alpha})$ |

Figure: Convergence rates of SMD. S. Krichene, W. Krichene, R. Dong, A. Bayen. In preparation.

[8] A. S. Nemirovsky and D. B. Yudin. *Problem complexity and method efficiency in optimization.*
Wiley-Interscience series in discrete mathematics. Wiley, 1983.
ISBN 9780471103455

# Regret bound for SMD

Main ingredient:

**Proposition**

*Assume* $D_\psi$ *bounded by* $D$ *and* $\mathbb{E}\,\|\hat{g}\|^2 \le G$*.*
*SMD method with* $(\eta_t)$*.* $\forall t_2 > t_1 \ge 0$ *and* $\mathcal{F}_{t_1}$*-measurable* $x$*,*

$$\sum_{\tau=t_1}^{t_2} \mathbb{E}\left[\left\langle g^{(\tau)}, x^{(\tau)} - x \right\rangle\right] \le \frac{\mathbb{E}\left[D_\psi(x, x^{(t_1)})\right]}{\eta_{t_1}} + D\left(\frac{1}{\eta_{t_2}} - \frac{1}{\eta_{t_1}}\right) + \frac{G}{2\ell_\psi}\sum_{\tau=t_1}^{t_2} \eta_\tau \quad (2)$$

# Distributed SMD with heterogeneous agents

- $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K$
- Agent $k$ updates $x^k \in \mathcal{X}_k$
- $D_{\psi^k}$ and $\eta_t^k$ depends on $k$

**Algorithm 4** DSMD Method with learning rates $(\eta_t^k)$ and divergences $D_{\psi^k}$

1: **for** $t \in \mathbb{N}$ **do**

2: $\quad \hat{g}^{(t)} \in \mathcal{F}_{t+1}$

3:

$$x_{\mathcal{P}_k}^{(t+1)} = \arg\min_{x_{\mathcal{P}_k} \in \mathcal{X}_k} \left\langle \hat{g}_{\mathcal{P}_k}^{(t)}, x_{\mathcal{P}_k} - x_{\mathcal{P}_k}^{(t)} \right\rangle + \frac{1}{\eta_t^k} D_{\psi^k}\left(x_{\mathcal{P}_k}, x_{\mathcal{P}_k}^{(t)}\right)$$

4: **end for**

# Routing game with heterogeneous populations

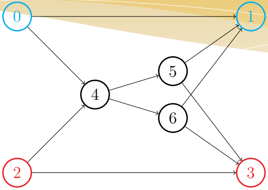



Figure: Example network

## Routing game with heterogeneous populations

Under unbiased noisy losses, with heterogeneous update rules with $\eta_t^k = \theta_k t^{-\alpha_k}$

$$\mathbb{E}\left[f(x^{(t)})\right] - f^\star = O\left(t^{-\min(\min_k \alpha_k, 1-\max_k \alpha_k)}\right)$$

where $f$ is the Rosenthal potential function



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# Simulations

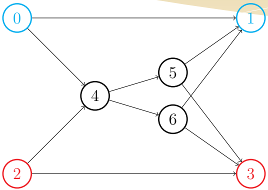



Figure: Example network

- Centered Gaussian noise on edges.
- Population 1: Hedge with $\eta_t^1 = t^{-0.1}$
- Population 2: Hedge with $\eta_t^2 = \frac{1}{2} t^{-0.5}$

# One realization

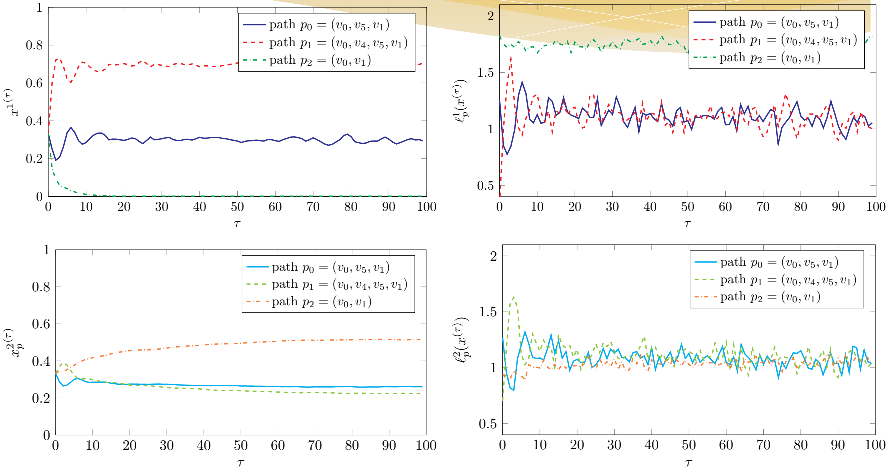



Figure: Population distributions and noisy path losses



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# In Expectation

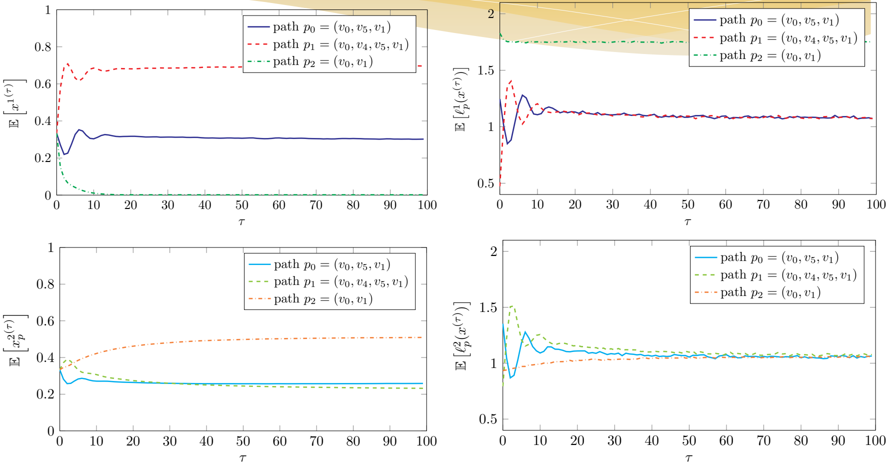



Figure: Population distributions and noisy path losses



FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS

# One realization

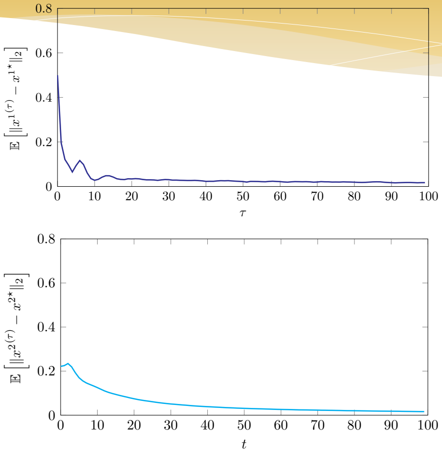



Figure: Expected distance to equilibrium

# Summary and Extensions

- How do players arrive at equilibrium?
  Any algorithm in the AREP or the DSMD class.
- How fast?
  Convergence rates for the DSMD class.
- Stability?
  Nash equilibria are stable for these dynamics [4]
- Robustness?
  Robust to unbiased perturbation, e.g. when losses are not known but estimated.

Extensions

- Provides a model of population dynamics for optimal control problems.
- Adapt to other problems, such as network consensus.

[4] Benjamin Drighès, Walid Krichene, and Alexandre Bayen. Stability of nash equilibria in the congestion game under replicator dynamics.
In *Conference on Decision and Control (CDC)*, 2014



FORCES
FOUNDATIONS OF RESILIENT
CYBER-PHYSICAL SYSTEMS

# Summary and Extensions

- How do players arrive at equilibrium?
  Any algorithm in the AREP or the DSMD class.
- How fast?
  Convergence rates for the DSMD class.
- Stability?
  Nash equilibria are stable for these dynamics [4]
- Robustness?
  Robust to unbiased perturbation, e.g. when losses are not known but estimated.

Extensions

- Provides a model of population dynamics for optimal control problems.
- Adapt to other problems, such as network consensus.

---

[4] Benjamin Drighès, Walid Krichene, and Alexandre Bayen. Stability of nash equilibria in the congestion game under replicator dynamics.
In *Conference on Decision and Control (CDC)*, 2014



FORCES
FOUNDATIONS OF RESILIENT
CYBER-PHYSICAL SYSTEMS

Thank you.

# References I


[1] Michel Benaïm. Dynamics of stochastic approximation algorithms. In *Séminaire de probabilités XXXIII*, pages 1–68. Springer, 1999.

[2] Avrim Blum, Eyal Even-Dar, and Katrina Ligett. Routing without regret: on convergence to nash equilibria of regret-minimizing algorithms in routing games. In *Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing*, PODC '06, pages 45–52, New York, NY, USA, 2006. ACM.

[3] Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games.* Cambridge University Press, 2006.

[4] Benjamin Drighès, Walid Krichene, and Alexandre Bayen. Stability of nash equilibria in the congestion game under replicator dynamics. In *Conference on Decision and Control (CDC)*, 2014.

[5] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004.

# References II


[6] James Hannan. Approximation to Bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–139, 1957.

[7] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 163–171. JMLR Workshop and Conference Proceedings, 2014.

[8] A. S. Nemirovsky and D. B. Yudin. *Problem complexity and method efficiency in optimization*. Wiley-Interscience series in discrete mathematics. Wiley, 1983. ISBN 9780471103455.

[9] William H Sandholm. Potential games with continuous player sets. *Journal of Economic Theory*, 97(1):81–108, 2001.

[10] Jörgen W Weibull. *Evolutionary game theory*. MIT press, 1997.




FORCES
FOUNDATIONS OF RESILIENT CYBER-PHYSICAL SYSTEMS