Stochastic control with input and state constraints: a relaxation technique to ensure feasibility

Luca Deori, Simone Garatti, Maria Prandini

Abstract— **We consider the problem of designing a finite-horizon control policy for a stochastic linear system subject to probabilistic constraints on both input and state variables. When the disturbance has unbounded support, a feasibility issue may arise due to the presence of the state constraint. In this paper, we address this issue by introducing a suitable relaxation of the original problem that ensures feasibility. The relaxation is such that the original state constraint is enforced whenever is possible; otherwise, the control that pushes the state closest to the constraint is chosen. This involves formulating a cascade of two chance-constrained optimization problems, which are tackled through a scenario-based randomized scheme expressly tailored to the problem at hand. The theoretical properties of the obtained solution are investigated and it is shown that randomization allows one to achieve computational tractability. The proposed approach finds immediate application to stochastic model predictive control.**

I. INTRODUCTION

Consider a discrete time stochastic linear system whose state $x_t \in \mathbb{R}^n$ evolves according to the equation

$$x_{t+1} = Ax_t + Bu_t + B_w w_t, \tag{1}$$

where $u_t \in \mathbb{R}^m$ is the control input and $w_t \in \mathbb{R}^{n_w}$, $n_w \leq n$, is a stochastic disturbance with known probability distribution $\mathbb{P}$ over a possibly unbounded support. Matrices A, B, and B_w have appropriate dimensions, and B_w is assumed to be full column rank.

Define

$$J = \mathbb{E}\left[\sum_{t=1}^{M} x_t^T Q_t x_t + \sum_{t=0}^{M-1} u_t^T R_t u_t\right], \tag{2}$$

where Q_t and R_t are symmetric and positive semi-definite matrices, M is the finite horizon length, and $\mathbb{E}$ denotes the expectation taken with respect to probability $\mathbb{P}$.
Our objective is to find a state feedback control policy that minimizes the finite-horizon average quadratic cost J, while accounting for constraints on the input and state variables. Given that the input u_t and state x_t are uncertain, since they both depend on the value taken by the stochastic disturbance w_t, constraints are formulated in probabilistic terms. More specifically, they are required to hold with a predefined (usually high) probability $1-\varepsilon$:

$$\mathbb{P}\{f(u_0,\dots,u_{M-1}) \leq \bar{u} \wedge g(x_1,\dots,x_M) \leq \bar{y}\} \geq 1-\varepsilon, \tag{3}$$

where $\varepsilon \in (0,1)$ is a user-chosen parameter, and $f : \mathbb{R}^{mM} \to \mathbb{R}^{p_u}$ and $g : \mathbb{R}^{nM} \to \mathbb{R}^{p_y}$ are convex and continuous vector-valued functions, $\bar{u} \in \mathbb{R}^{p_u}$ and $\bar{y} \in \mathbb{R}^{p_y}$, and the inequalities are to be interpreted component-wise.

Typically, constraints on the input and state variables are represented by bounds on their norm, e.g.,

$$f(u_0,\dots,u_{M-1}) = \begin{bmatrix} \|u_0\|_\infty \\ \vdots \\ \|u_{M-1}\|_\infty \end{bmatrix} \leq \bar{u},$$

$$g(x_1,\dots,x_M) = \begin{bmatrix} \|Cx_1\|_\infty \\ \vdots \\ \|Cx_M\|_\infty \end{bmatrix} \leq \bar{y}.$$

To the purpose of control design, the following disturbance feedback parametrization is adopted:

$$u_i = \gamma_i + \sum_{j=0}^{i-1} \theta_{i,j} w_j \qquad i = 0\dots M-1, \tag{4}$$

where the open-loop terms $\gamma_i \in \mathbb{R}^m$ and the disturbance feedback gains $\theta_{i,j} \in \mathbb{R}^{m\times n_w}$ are design parameters to be optimized. The linear parametrization in (4) ensures that both the input u_t and the state x_t are affine functions of γ_i and $\theta_{i,j}$, and, hence, the cost J in (2) is quadratic. In the following, we shall assume that J is strictly convex as a function of the control design parameters.[1]

Note that, since B_w is full column rank with pseudo-inverse $B_w^\dagger$, the values taken by the stochastic disturbance w_t can be recovered from the state measurements

$$w_t = B_w^\dagger(x_{t+1} - Ax_t - Bu_t), \tag{5}$$

which shows that the disturbance feedback control policy in (4) is indeed a state feedback control policy. Interestingly, (4) is equivalent to a control policy affine in the state, and one can recover the optimal unconstrained LQ-control law by properly setting Γ and Θ, [1].

Note that the possibly unbounded disturbance w_t enters additively on the state (see (1)). If g grows unbounded when the state increases, as it is the case for norms, then the feasibility domain as given by (3) may be void, depending on whether or not the value of $\bar{y}$ is compatible with the system dynamics, the disturbance characteristics, the input constraint, and the allowed violation probability ε. This is

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[1] A sufficient condition for strict convexity to hold is that matrices $\mathbf{R}$ and $\mathbb{E}[\mathbf{w}\mathbf{w}^T]$ are positive definite.

the issue addressed in the present paper, where a suitable relaxation of the problem

$$\min_{\gamma_i,\theta_{i,j}} J \qquad (6)$$
$$\text{s.t. } \mathbb{P}\{f(u_0,\ldots,u_{M-1}) \leq \bar{u} \wedge g(x_1,\ldots,x_M) \leq \bar{y}\} \geq 1-\varepsilon$$

is proposed to recover feasibility. The relaxation is conceived so as to enforce the original state bound $\bar{y}$ whenever is possible, while, otherwise, the smallest feasible state bound is first determined and then imposed so as to keep the state as close as possible to the desired domain. This translates into the cascade of two chance-constrained optimization problems, which is hard to solve. A randomized resolution scheme to enhance computational tractability is proposed and a theoretical analysis of its properties is provided. The findings of this paper have immediate implications on the recursive feasibility of the implementation of the proposed control strategy over a receding horizon as done in model predictive control.

A. Paper structure

The rest of the paper unfolds as follows. After some bibliographical remarks in Section I-B, we introduce some compact notation in Section I-C. In Section II, the problem relaxation is formally introduced, while its algorithmic solution based on randomization is given in Section III. In this section, the theoretical properties of the obtained solution are also discussed. A numerical example is finally provided in Section IV.

B. Bibliographical remarks

Alternative approaches to tackle problems with both input and state constraints and where the disturbance has unbounded support have been proposed in [2], [3], [4], [5], [6], [7], [8], [9], [10]. In [2], [6], [7], state constraints are replaced by a penalization term accounting for the state constraint violation so as to avoid infeasibility. In [3], [5], [8], [9], [10], an analytic convex relaxation of chance constraints is adopted, while in [4] the problem is reduced to one with bounded disturbance by suitably cutting the tails of the disturbance distribution. In all cases, the disturbance is assumed to be a sequence of i.i.d. (independent and identically distributed) random variables. Many approaches also assume that the disturbance has a Gaussian distribution, [2], [3], [4], [5], [8], [10].

This paper differs from these approaches in that a randomized-based solution is considered, which allows us to drop the independence and Gaussianity assumptions. Indeed, randomized methods have been recently developed to address design in the presence of uncertainty, making solvable problems that were otherwise deemed computationally intractable, [11]. This paper differs from our previous conference contribution [12] where either a term penalizing state constraint violation is added to the cost or a certain predefined admissible deterioration of the system performance is introduced while relaxing the state constraints.

Other approaches to constrained stochastic control for system (1) based on randomized techniques have been proposed in [13], [14], [15] under the assumption, however, that the noise has bounded support, or in [16] considering input constraints only.

C. Compact notation

In order to ease the notation we define:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix}.$$

Then, it is easy to show that

$$\mathbf{x} = \mathbf{F}x_0 + \mathbf{G}\mathbf{u} + \mathbf{H}\mathbf{w} \qquad (7)$$

where matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are given by

$$\mathbf{F} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^M \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} B & 0_{n\times m} & \cdots & 0_{n\times m} \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{n\times m} \\ A^{M-1}B & \cdots & AB & B \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} B_w & 0_{n\times n_w} & \cdots & 0_{n\times n_w} \\ AB_w & B_w & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{n\times n_w} \\ A^{M-1}B_w & \cdots & AB_w & B_w \end{bmatrix}.$$

Similarly, by setting

$$\Gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{M-1} \end{bmatrix} \Theta = \begin{bmatrix} 0_{m\times n_w} & 0_{m\times n_w} & \cdots & 0_{m\times n_w} \\ \theta_{1,0} & 0_{m\times n_w} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{m\times n_w} \\ \theta_{M-1,0} & \cdots & \theta_{M-1,M-2} & 0_{m\times n_w} \end{bmatrix},$$

the disturbance feedback policy (4) can be rewritten in the following compact form

$$\mathbf{u} = \Gamma + \Theta\mathbf{w}.$$

By plugging this expression into (7), we can make explicit the affine dependence of $\mathbf{x}$ on the design parameters Γ and Θ:

$$\mathbf{x} = \mathbf{F}x_0 + \mathbf{G}\Gamma + (\mathbf{G}\Theta + \mathbf{H})\mathbf{w}$$

As for the average quadratic cost (2), we can introduce suitable block diagonal matrices $\mathbf{Q}$ and $\mathbf{R}$ and write it as follows

$$J = \mathbb{E}\left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right].$$

This expression can be made explicit as a quadratic function of Γ and Θ. Derivations are straightforward and omitted here.

Finally, if we collect in λ the nonzero components of Γ and Θ, then, we can adopt the following notations

$$\mathbf{u} = \mathbf{u}(\mathbf{w};\lambda),\ \mathbf{x} = \mathbf{x}(\mathbf{w};\lambda),\ J = J(\lambda),$$

which point out the dependence of input, state, and cost on the optimization vector λ and the disturbance realization $\mathbf{w}$.

II. Problem relaxation to ensure feasibility

The idea developed to guarantee feasibility is as follows. We replace $\bar{y}$ in (3) with an optimization variable $h \in \mathbb{R}^{p_y}$ to be minimized component-wise. This way, the state constraint is always feasible since by taking h large enough it becomes ineffective. However, in no way taking a too large h is a good choice and we see that the presence of the new optimization variable h requires to handle two typically conflicting objectives: the minimization of the bound on the state so as to satisfy the original state bound in (3), and the minimization of the cost function $J(\lambda)$ that represents the performance of the system.

We deal with these different objectives by means of a two-step approach whose goal is to stay as close as possible to the original problem. In the first step we focus on the bound on the state constraint: the optimization variable h is minimized and an additional constraint is enforced so as to ensure that h gets not smaller than the original bound $\bar{y}$. Then, in the second step, we focus on the minimization of the cost function $J(\lambda)$, where the optimal h obtained in the first step is taken as bound for the state constraint. That is,

$$\min_{\lambda,h} h^T T h \tag{8a}$$
$$\text{s.t.} \begin{cases} \mathbb{P}\{f(\mathbf{u}(\mathbf{w};\lambda)) \le \bar{u} \wedge g(\mathbf{x}(\mathbf{w};\lambda)) \le h\} \ge 1-\varepsilon \\ h \ge \bar{y} \end{cases} ;$$

$$\min_{\lambda} J(\lambda) \tag{8b}$$
$$\text{s.t. } \mathbb{P}\{f(\mathbf{u}(\mathbf{w};\lambda)) \le \bar{u} \wedge g(\mathbf{x}(\mathbf{w};\lambda)) \le h^o\} \ge 1-\varepsilon,$$

where h^o is the optimal value for h obtained in (8a).

In (8), T is a positive definite matrix that can assign a different importance to the different components of h.

The idea behind the cascade of problems in (8) is as follows. When the original problem (6) is not feasible, the minimization of h in (8a) in the first step allows one to find the smallest bound on the state that preserves feasibility; instead, when (6) is feasible, h^o can be taken as $\bar{y}$. As a matter of fact, the constraint $h \ge \bar{y}$ is introduced in (8a) to avoid excessively conservative solutions where the bound on the state constraint is smaller than $\bar{y}$. Note that the objective function in (8a) does not depend on λ and it maybe that the same h^o is attained for different values of λ, each one possibly achieving a different value of the cost function $J(\lambda)$. This extra degree of freedom is exploited in the second step to optimize the performance of the system. Note that in (8b) feasibility is not an issue anymore, since the bound h^o computed in the first step is adopted in the probabilistic constraint. The overall solution returned by the cascade of problems (8) is a pair (λ^o, h^o), where λ^o determines the control action to be implemented and h^o is the probabilistically guaranteed bound for the state constraint. This h^o, once computed, can be inspected for comparison with the original $\bar{y}$.

III. Scenario-based resolution scheme

Problems (8a) and (8b) are, in general, hard to solve because of the presence of a probabilistic constraint, which can be non-convex in spite of the convexity of f and g. In order to enhance computational tractability, some approximation has to be accepted. Here, we resort to a randomized scheme that is in the line of the so-called scenario approach, [17], [18], [19], [20]. In this scheme, the returned solution comes accompanied by precise guarantees about feasibility for the original probabilistic constraint in problems (8a).

The idea of the scenario scheme is quite simple: N disturbance realizations are extracted according to the underlying probability distribution, say

$$\mathbf{w}^{(1)} = \begin{bmatrix} w_0^{(1)} & w_1^{(1)} & \dots & w_{M-1}^{(1)} \end{bmatrix}$$
$$\mathbf{w}^{(2)} = \begin{bmatrix} w_0^{(2)} & w_1^{(2)} & \dots & w_{M-1}^{(2)} \end{bmatrix}$$
$$\vdots$$
$$\mathbf{w}^{(N)} = \begin{bmatrix} w_0^{(N)} & w_1^{(N)} & \dots & w_{M-1}^{(N)} \end{bmatrix};$$

then, in each optimization problem, the probabilistic constraint is replaced by N non probabilistic constraints, those obtained in correspondence of the seen disturbance realizations. Precisely, the scenario version of (8a) and (8b) consists in the following cascade of optimization problems:

$$\begin{aligned} \min_{\lambda,h} \quad & h^T T h \\ \text{s.t.} \quad & f(\mathbf{u}(\mathbf{w}^{(k)};\lambda)) \le \bar{u}, \\ & g(\mathbf{x}(\mathbf{w}^{(k)};\lambda)) \le h, \quad k = 1,\dots,N, \\ & h \ge \bar{y}; \end{aligned} \tag{9a}$$

$$\begin{aligned} \min_{\lambda} \quad & J(\lambda) \\ \text{s.t.} \quad & f(\mathbf{u}(\mathbf{w}^{(k)};\lambda)) \le \bar{u}, \\ & g(\mathbf{x}(\mathbf{w}^{(k)};\lambda)) \le h^\star, \quad k = 1,\dots,N, \end{aligned} \tag{9b}$$

where $h^\star$ is the optimal value of h obtained in (9a).

Note that (9a) and (9b) are convex problems with a finite number of constraints, so that they can be solved by resorting to standard solvers for convex optimization, that is, randomization has led us back to computational tractability. Moreover in this case, by convexity of constraints, since $h^T T h$ is strictly convex with respect to h, the value $h^\star$ is uniquely determined by (9a), and, similarly, the solution to (9b), say $\lambda^\star$, is unique.

The overall solution to the cascade of problems (9) is defined as $(\lambda^\star, h^\star)$, and, (9) can be seen as an empirical counterpart of (8). All comments regarding the interpretation of (8) applies *mutatis mutandis* to (9). Note eventually that the pair $(\lambda^\star, h^\star)$ is feasible and optimal for (9), so that the second optimization problem (9b) can be thought of as a tie break rule to choose among the possible multiple solutions of the first optimization problem (9a) the one that minimizes $J(\lambda)$.

A. Chance-constraint feasibility of the scenario solution

Using $(\lambda^\star, h^\star)$ in place of (λ^o, h^o) is the price we have to pay in order to enhance computational tractability. However, one main question arises about the feasibility of $(\lambda^\star, h^\star)$ for the probabilistic constraint

$$\mathbb{P}\{f(\mathbf{u}(\mathbf{w};\lambda)) \le \bar{u} \wedge g(\mathbf{x}(\mathbf{w};\lambda)) \le h\} \ge 1-\varepsilon, \qquad (10)$$

so as to establish a link between the obtained scenario-based solution and the original problem (8). The theory of the scenario approach has mainly dealt with this question, showing that the answer is indeed affirmative with high confidence in a number of contexts, provided that N is big enough, [17], [18], [19], [21], [22]. The best available result is that of [19] which largely improves over previous achievements. However, to date the result of [19] has been proven only for scenario optimization programs whose solution is determined by a specific tie break rule which may differ from the one corresponding to the cascade of problems in (9) (see point 5 in Section 2.1 of [19]). The following theorem provides the extension of the result obtained in [19] to the present setup where a cascade of problems is considered.

Theorem 1: Let $\beta \in (0,1)$ be a user-chosen confidence parameter. If the number of extracted disturbance realizations N is chosen so as to satisfy

$$\sum_{i=0}^{d-1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \le \beta, \qquad (11)$$

where d is the dimensionality of (λ, h), then it holds with confidence at least $1-\beta$ that

$$\mathbb{P}\{f(\mathbf{u}(\mathbf{w};\lambda^\star)) \le \bar{u} \,\wedge\, g(\mathbf{x}(\mathbf{w};\lambda^\star)) \le h^\star\} \ge 1-\varepsilon.$$

Due to space limitation the proof is omitted. It is, however, available in [23] and on request from the authors.

In words, the theorem says that the scenario-based solution $(\lambda^\star, h^\star)$ is feasible for the probabilistic constraint (10) with confidence at least $1-\beta$. The fact that the guarantee is provided with some confidence only is intrinsically so, because $(\lambda^\star, h^\star)$ depends on the random extraction of $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(N)}$, and β keeps into account the possibility of seeing an anomalous sample $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(N)}$ which is not representative enough. However, it has to be noted that, as shown in [24], the N satisfying (11) scales logarithmically with $1/\beta$, so that small values of β, like 10^{-5} or even 10^{-7}, can be forced in without affecting N too much. With such small values for β, one can rely on the fact that $(\lambda^\star, h^\star)$ is feasible for the probabilistic constraint (10), and hence for (8), beyond any reasonable doubts.

IV. Numerical Example

In this section the proposed approach is applied to a numerical example taken from [10]. We consider a mechanical system composed by 4 masses and 4 springs as depicted in Fig. 1.

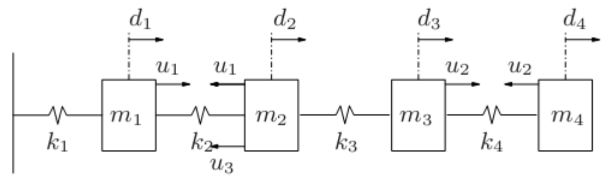


Fig. 1: Scheme of the mechanical system.

The state of the system is given by the displacement of masses with respect to their nominal positions, and by their derivatives, i.e. $x = [d_1,\ d_2,\ d_3,\ d_4,\ \dot{d}_1,\ \dot{d}_2,\ \dot{d}_3,\ \dot{d}_4]^T$. The initial condition is $x_0 = [10,\ -10,\ 10,\ -10,\ 0,\ 0,\ 0,\ 0]^T$. The control inputs are the forces u_1, u_2, u_3 acting on the masses shown in Fig. 1. All the masses and the stiffness coefficients are set to 1, and the system is written as in (1) by discretizing it under the assumption that the input is kept constant in the interval $[t,\ t+T_s)$, with $T_s = 1$ s. We also suppose that the masses displacements and velocities are affected by a stochastic additive disturbance, which after discretization leads to $w \sim WGN(0,\ I_4)$ and $B_w = [0.5I_4\ I_4]^T$.

The objective is to keep the masses positions as close as possible to the nominal ones, while, at the same time, a requirement on the maximum speed of the masses has to be satisfied. To this purpose, the weight matrices in the cost function J are set so as to penalize deviations from the nominal positions:

$$Q = \begin{bmatrix} I_4 & 0_{4\times4} \\ 0_{4\times4} & 0_{4\times4} \end{bmatrix} \quad R = 10^{-6} I_3,$$

and the following constraints are enforced on the speeds of the masses:

$$\|Cx_i\|_\infty \le \bar{y}_i, \qquad i = 1, \dots, M,$$

where $C = [0_{4\times4}\ I_4]$, $\bar{y}_i = 10$, $i = 1, \dots, M$ and $M = 8$ is the considered time horizon. The constraint is enforced in probability with $\varepsilon = 0.1$. No input constraints are instead imposed.

Following the approach in Section II, the bounds $\bar{y}_i$, $i = 1, \dots, M$ were replaced by the optimization variables h_i, $i = 1, \dots, M$. We set $\beta = 10^{-6}$, returning $N = 4614$ according to (11), and the cascade of problems (9) with $T = I_8$ was then solved to obtain the scenario-based solution.

It turned out that the original $\bar{y}$ was not feasible as for the first 2 time instants. Indeed, the smallest feasible bounds obtained from the resolution of (9a) were $h_1^\star = 11.62$, $h_2^\star = 11.08$ and $h_i^\star = \bar{y}_i$, $i = 3, \dots, M$. The optimal scenario-based control policy obtained from problem (9b) achieved a cost $J(\lambda^\star) = 2305.55$. A Monte Carlo verification revealed that the probabilistic constraint (10) was satisfied by $(\lambda^\star, h^\star)$ as guaranteed by Theorem 1.

For the sake of comparison, the obtained scenario control policy was tested against a finite horizon LQ controller. In order to somehow account for the speed requirement, in the design of the LQ controller, the cost function was modified so as to assign a penalization on the speed of the masses:

$$J_{LQ} = \mathbb{E}[\mathbf{x}^T \mathbf{Q}_{LQ}\mathbf{x} + \mathbf{u}^T \mathbf{R}_{LQ}\mathbf{u}]$$

$$Q_{LQ} = \begin{bmatrix} q_J I_4 & 0_{4\times 4} \\ 0_{4\times 4} & q_{\bar{y}} I_4 \end{bmatrix} \quad R_{LQ} = 10^{-6} I_3$$

where the weights q_J and $q_{\bar{y}}$ permits one to tune the relative importance between positions and velocities in the cost function.

The performances of the obtained scenario-based control policy and of the LQ controller for various choices of q_J and $q_{\bar{y}}$ are compared in Table I – where the achieved cost J and the actual probability of violation $\varepsilon_{\bar{y}}$ of the original constraint with $\bar{y}$, as computed via Monte Carlo simulations, are reported for all the approaches – and in Fig. 2 – where the cumulative probability distributions of $\|Cx_i\|_\infty$, $i = 1,\dots,M$, again computed via Monte Carlo simulations, are depicted for all the approaches.

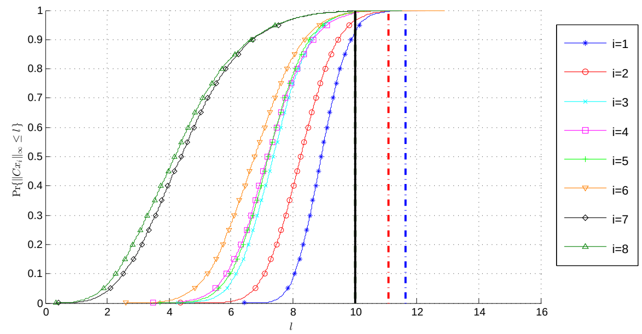
TABLE I

q_J	$q_{\bar{y}}$	Approach	J	$\varepsilon_{\bar{y}}$
-	-	Scenario based	2305.55	0.1248
1	0	LQ	126.44	1
0	1	LQ	4347.20	0.9724
0.2	9	LQ	2318.50	0.9960

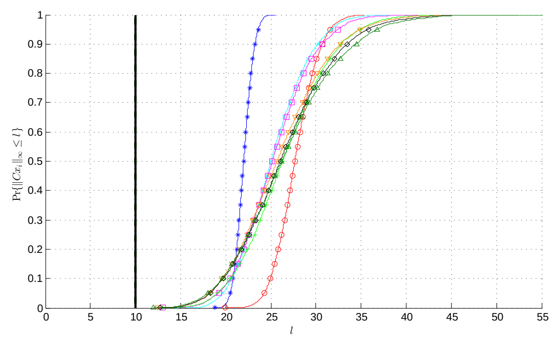
As it appears, the proposed scenario-based approach achieves a good trade-off between J and $\varepsilon_{\bar{y}}$ and, in particular, this latter, though not equal to the required 0.1 value (which was eventually unfeasible), is very close to 0.1 since in the first step the $h_i^\star$ was pushed towards $\bar{y}_i$ as much as possible.

When the LQ controller is designed accounting for the mass displacements only ($q_J = 1$ $q_{\bar{y}} = 0$), the cost function J turns out to be much improved, but the speed limit $\bar{y}$ is violated by a huge extent, see in particular Fig. 2(b). When on the contrary the LQ controller is designed accounting for speeds only while displacements are neglected, the cost function J is worse than the one obtained by the scenario-based solution, while $\varepsilon_{\bar{y}}$ is still very high as compared to 0.1. In particular, as shown in Fig. 2(c), the constraint is significantly violated in the first time step, while, in the other time steps, the velocity is excessively reduced with respect to the allowed limit $\bar{y}$. Also in the third case, where $q_J = 0.2$ and $q_{\bar{y}} = 9$ are chosen so as to make the LQ controller achieving a cost J close to the one obtained by the scenario-based solution, the same kind of violation of the constraint as in the previous case is obtained, see Fig. 2(d).

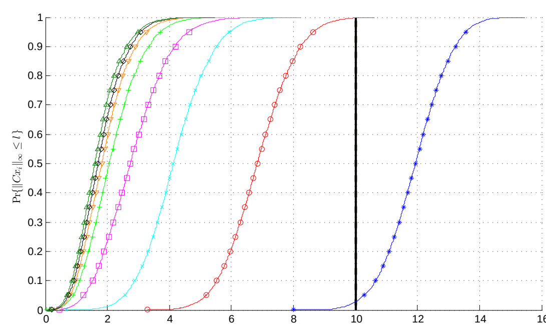
The behaviors of the different controllers can be appreciated also analysing the state trajectories reported in



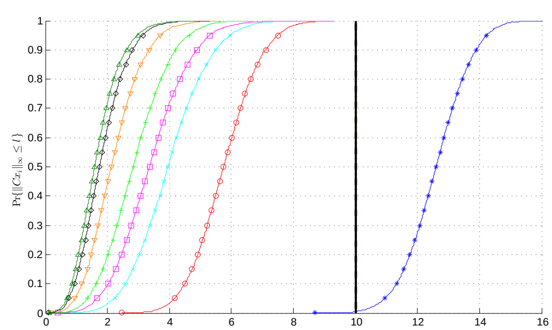
(a) Scenario based solution, $h_1^\star$ blue dash-dotted line, $h_2^\star$ red dash-dotted line.



(b) LQ controller $q_J = 1$, $q_{\bar{y}} = 0$



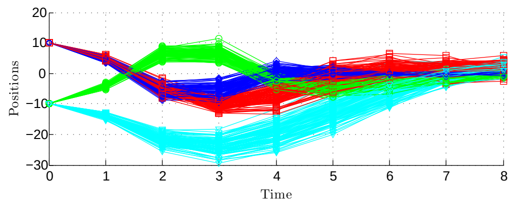
(c) LQ controller $q_J = 0$, $q_{\bar{y}} = 1$



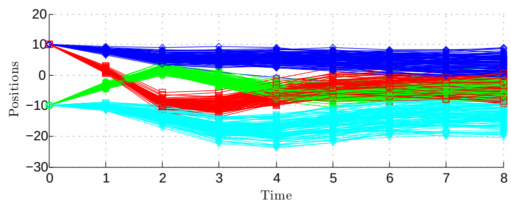
(d) LQ controller $q_J = 0.2$, $q_{\bar{y}} = 9$

Fig. 2: Cumulative probability distributions of $\|Cx_i\|_\infty$, $i = 1,\dots,8$ for the scenario based solution and for the LQ controllers; $\bar{y}$ is represented by the marked vertical solid line.

Fig. 3 (displacements) and Fig. 4 (velocities) for 100 new disturbance realizations. The scenario-based controller, by exploiting the allowed speed, is able to steer the masses close to their nominal positions. Instead, although the LQ controller violates the constraint in the first time step, in the other steps it conservatively keeps the speeds too small, so that the masses are not steered to the nominal position.

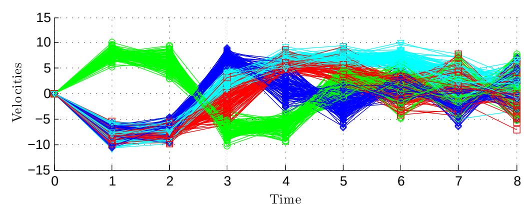


(a) Scenario-based controller

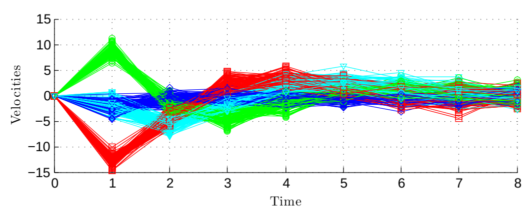


(b) LQ controller $q_J = 0.2$, $q_{\bar{y}} = 9$

Fig. 3: Displacements of the masses: d_1 (blue diamonds), d_2 (green circles), d_3 (red squares), d_4 (cyan triangles).



(a) Scenario-based controller



(b) LQ controller $q_J = 0.2$, $q_{\bar{y}} = 9$

Fig. 4: Velocities of the masses: $\dot{d}_1$ (blue diamonds), $\dot{d}_2$ (green circles), $\dot{d}_3$ (red squares), $\dot{d}_4$ (cyan triangles).

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