Learning Dynamics, Estimation and Control In Congestion Games

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Learning dynamics in the routing game

- Routing games model congestion on networks.
- Transportation, communication networks
- Nash equilibrium quantifies efficiency of network in steady state.

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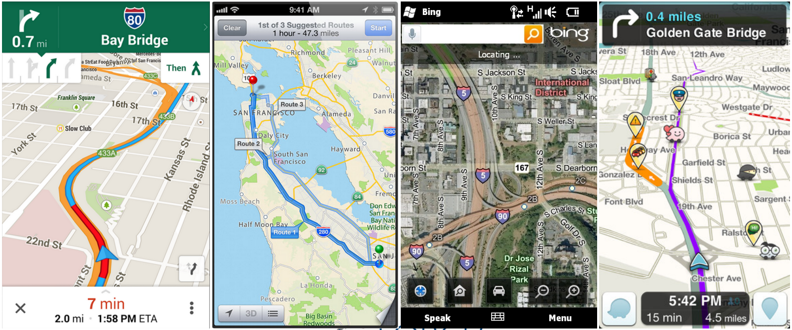

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Outline

Congestion games

Routing game

- Player k drives from source to destination node
- Chooses path from $\mathcal{A}_k$
- Mass of players on each edge determines cost on that edge.

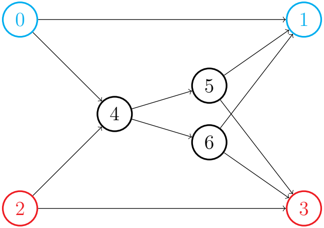

Figure : Routing game

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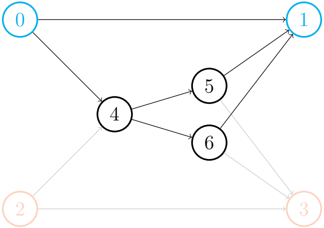

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Online learning model

Algorithm 1 *

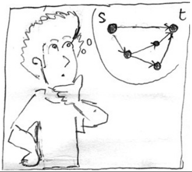
for $t \in \mathbb{N}$ **do**
    Play $p \sim x_k^{(t)}$
    Discover $\ell_k^{(t)}$
    Update

$$x_k^{(t+\mathbf{1})} = \arg\min_{x \in \Delta^{\mathcal{A}_k}} \left\langle \ell_k^{(t)}, x_k \right\rangle + \frac{1}{\eta_k^{(t)}} D_{\mathbf{KL}}(x_k^{(t)}, x_k)$$
$$\propto e^{-\eta_k^{(t)} \ell_k^{(t)}}$$

end for

Online Learning Model

$x_{\mathbf{1}}^{(t)} \in \Delta^{\mathcal{A}_{\mathbf{1}}}$



[4]Walid Krichene, Syrine Krichene, and Alexandre Bayen. Efficient Bregman projections onto the simplex.
In *54th IEEE Conference on Decision and Control (CDC)*, Osaka, Japan, 2015

Online learning model

Algorithm 2 *

for $t \in \mathbb{N}$ **do**
  Play $p \sim x_k^{(t)}$
  Discover $\ell_k^{(t)}$
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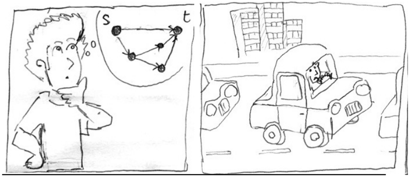
end for

Online Learning Model

$x_{\mathbf{1}}^{(t)} \in \Delta^{\mathcal{A}_{\mathbf{1}}}$ Sample $p \sim x_{\mathbf{1}}^{(t)}$



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Algorithm 3 *

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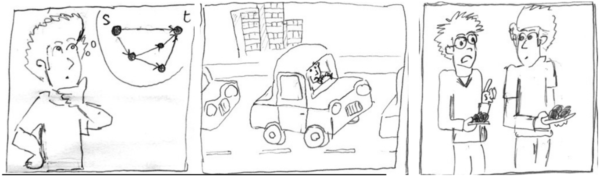
$$\propto e^{-\eta_k^{(t)} \ell_k^{(t)}}$$

end for

Online Learning Model

$x_{\mathbf{1}}^{(t)} \in \Delta^{\mathcal{A}_{\mathbf{1}}}$ Sample $p \sim x_{\mathbf{1}}^{(t)}$ Discover $\ell_{\mathbf{1}}^{(t)}$



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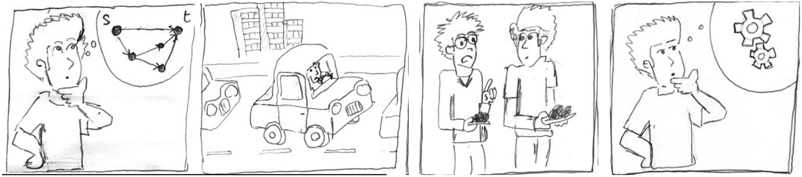
$$\propto e^{-\eta_k^{(t)} \ell_k^{(t)}}$$

end for

Online Learning Model

$x_{\mathbf{1}}^{(t)} \in \Delta^{\mathcal{A}_{\mathbf{1}}}$ | Sample $p \sim x_{\mathbf{1}}^{(t)}$ | Discover $\ell_{\mathbf{1}}^{(t)}$ | Update $x_{\mathbf{1}}^{(t+\mathbf{1})}$



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Estimation of Hedge Dynamics

Control under Hedge dynamics

Loss Function Imputation

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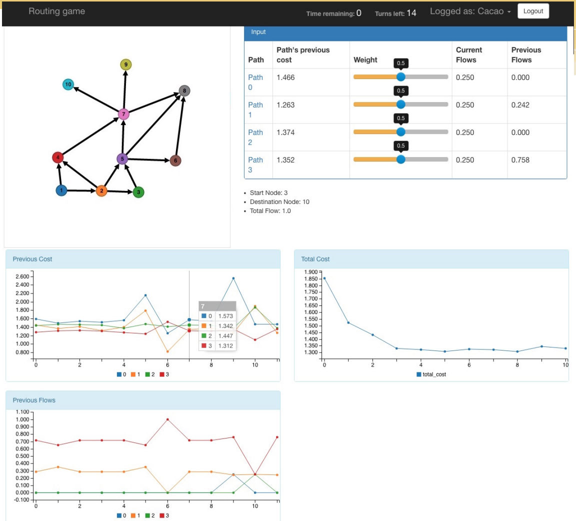
Estimation of learning dynamics



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Estimation of learning dynamics

- We observe a sequence of player decisions $(\bar{x}^{(t)})$ and losses $(\bar{\ell}^{(t)})$.
- Can we fit a model of player dynamics?

[6]Kiet Lam, Walid Krichene, and Alexandre Bayen. On learning how players learn: Estimation of learning dynamics in the routing game.
In *7th International Conference on Cyber-Physical Systems (ICCPS)*, 2016

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Mirror descent model

Estimate the learning rate in the mirror descent model

$$x^{(t+1)}(\eta) = \arg\min_{x \in \Delta^{\mathcal{A}_k}} \left\langle \bar{\ell}^{(t)}, x \right\rangle + \frac{1}{\eta} D_{KL}(x, \bar{x}^{(t)})$$

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Then $d(\eta) = D_{KL}(\bar{x}^{(t+1)}, x^{(t+1)}(\eta))$ is a convex function. Can minimize it to estimate $\eta_k^{(t)}$.

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[6]Kiet Lam, Walid Krichene, and Alexandre Bayen. On learning how players learn: Estimation of learning dynamics in the routing game.
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Results

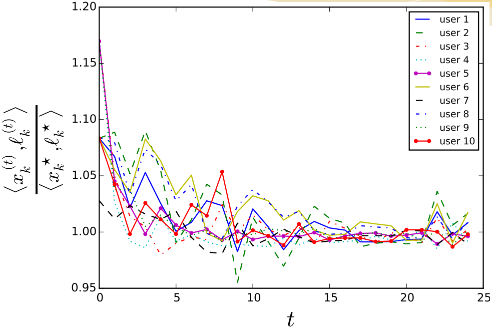

Figure : Costs of each player (normalized by the equilibrium cost)

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Results

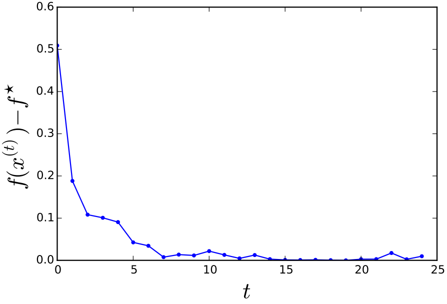

Figure : Potential function $f(x^{(t)}) - f^\star$.

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Results

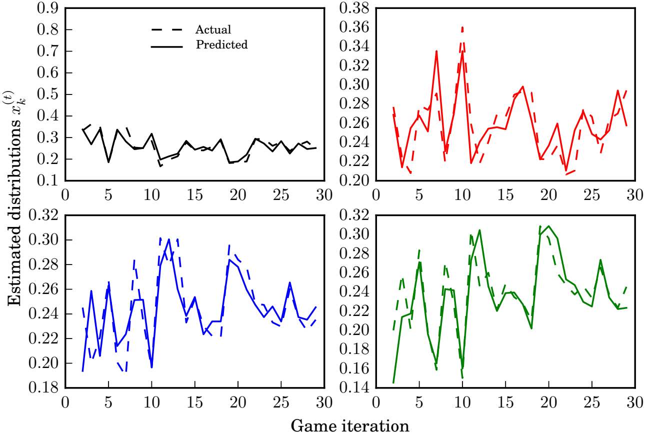

Figure : Estimated Vs. actual distribution.

Results

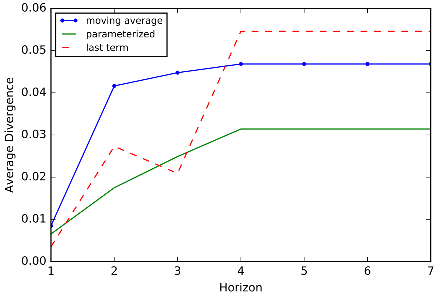

Figure : Average KL divergence between predicted distributions and actual distributions, as a function of the prediction horizon h.

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Optimal routing with learning dynamics

Assumptions

- A central authority has control over a fraction of traffic: $u^{(t)} \in \alpha_1 \Delta^{\mathcal{A}_1} \times \cdots \times \alpha_K \Delta^{\mathcal{A}_K}$
- Remaining traffic follows learning dynamics: $x^{(t)} \in (1-\alpha_1)\Delta^{\mathcal{A}_1} \times \cdots \times (1-\alpha_K)\Delta^{\mathcal{A}_K}$

Optimal routing under selfish learning constraints

$$\begin{aligned} \text{minimize}_{u^{(1:T)}, x^{(1:T)}} \quad & \sum_{t=1}^{T} J(x^{(t)}, u^{(t)}) \\ \text{subject to} \quad & x^{(t+1)} = u(x^{(t)} + u^{(t)}, \ell(x^{(t)} + u^{(t)})) \end{aligned}$$

[5]Walid Krichene, Milena Suarez, and Alexandre Bayen. Optimal routing under hedge response.
Transactions on Control of Networked Systems (TCNS), 2017

Solution methods

- Greedy method: Approximate the problem with a sequence of convex problems.

$$\text{minimize}_{u^{(t)}} J(u(x^{(t-1)}, u^{(t-1)}), u^{(t)})$$

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$$\text{minimize}_{u^{(t)}} J(u(x^{(t-1)}, u^{(t-1)}), u^{(t)})$$

- Mirror descent with the adjoint method.

Adjoint method

$$\begin{aligned} \text{minimize}_{u} \quad & J(u, x) \\ \text{subject to} \quad & H(x, u) = 0 \end{aligned}$$

equivalent to

$$\text{minimize} \quad J(u, X(u))$$

Then perform mirror descent on this function of u.

A simple example

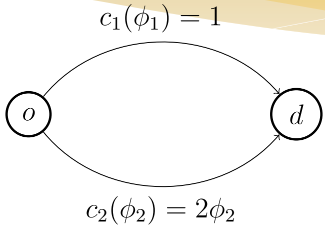

Figure : Simple Pigou network used for the numerical experiment.

- Social optimum: $(\frac{3}{4}, \frac{1}{4})$
- Nash equilibrium $(\frac{1}{2}, \frac{1}{2})$
- Control over $\alpha = \frac{1}{2}$ of traffic

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A simple example

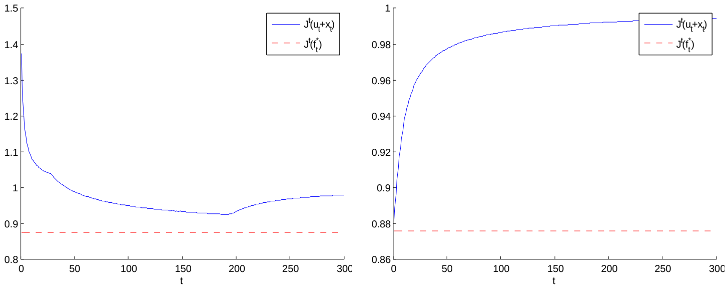

Figure : Social cost $J^{(t)}$ over time induced by adjoint solution (left) and the greedy solution (right). The dashed line shows the social optimal allocation.

A simple example

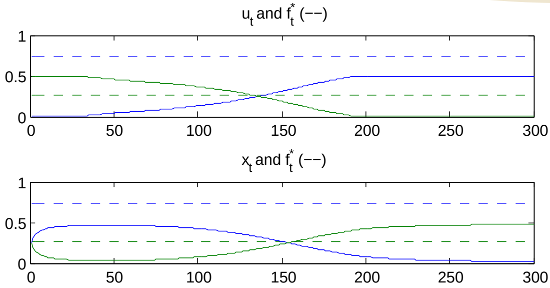

Figure : Adjoint controlled flows (top), selfish flows (bottom). The green lines correspond to the top path, and the blue lines to the bottom path. The dashed lines show the social optimal flows $x^{SO} = (\frac{3}{4}, \frac{1}{4})$.

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Application to the L.A. highway network

- Simplified model of the L.A. highway network.
- Cost functions uses the B.P.R. function, calibrated using the work of [8].

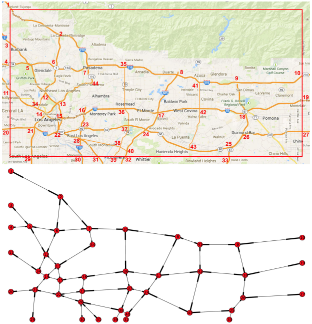

Figure : Los Angeles highway network.

[8]J. Thai, R. Hariss, and A. Bayen. A multi-convex approach to latency inference and control in traffic equilibria from sparse data.
In *American Control Conference (ACC), 2015*, pages 689–695, July 2015

Application to the L.A. highway network

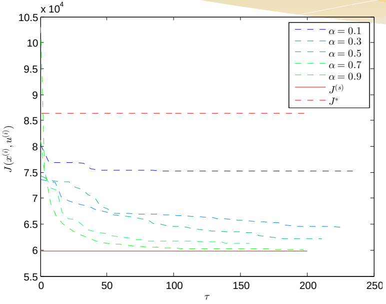

Figure : Average delay without control (dashed), with full control (solid), and different values of α.

Outline

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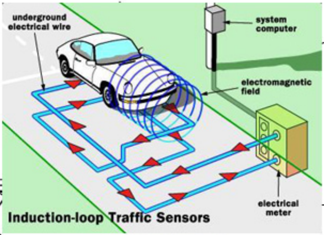
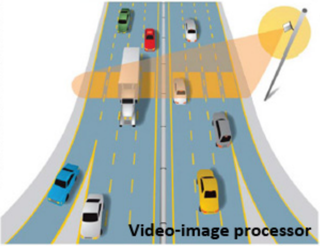
Motivation





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Framework

VI($\mathcal{K}, F$) consists in finding $\mathbf{x} \in \mathcal{K}$, *i.e.*, $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq 0$, such that

$$F(\mathbf{x})^T(\text{ by } - \mathbf{x}) \geq 0, \forall \text{ by } \in \mathcal{K} \tag{1}$$

Examples of parametric VI's VI($\mathcal{K}(\mathbf{p}), F(\cdot, \mathbf{p})$):

- Routing game with fixed latency functions and variable demand:

$$\ell(\mathbf{x})^T(\text{ by } - \mathbf{x}) \geq 0 \quad \forall \text{ feasible path flow by for demand } \mathbf{d}(\mathbf{p}) \tag{2}$$

- Parametric convex optimization:

$$\min\ f(\mathbf{x}, \mathbf{p}) \text{ s.t. } \mathbf{x} \in \mathcal{K}(\mathbf{p}) \quad \Longleftrightarrow \quad \text{VI}(\mathcal{K}(\mathbf{p}), \nabla f(\cdot, \mathbf{p})) \tag{3}$$

- Controller fitting, consumer behavior etc. [2, 1]

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[2]A. Keshavarz, Y. Wang, and S. Boyd. Imputing a convex opjective function.
In *IEEE International Symposium on Intelligent Control (ISIC)*, 2011

[1]Dimitris Bertsimas, Vishal Gupta, and Ioannis Paschalidis. Data-driven estimation in equilibrium using inverse optimization.
Math. Program., pages 595–633, 2015

Problem statement

Inputs:

- parametric polyhedron $\{\mathcal{K}(\mathbf{p})\}_{\mathbf{p}}$
- parametric observation process $g(\cdot, \mathbf{p}) : \mathbb{R}^n \to \mathbb{R}^q$
- N observations of equilibria $\mathbf{z}^{(j)} := g(\mathbf{x}^{(j)}, \mathbf{p}^{(j)}) + \mathbf{w}^{(j)},\ j \in [N]$

Objective:

Impute parametric map $F(\cdot, \mathbf{p})$ and decision vectors $\mathbf{x}^{(j)}$ such that

(a) $\mathbf{x}^{(j)}$ is an approximate solution to $\mathrm{VI}(\mathcal{K}(\mathbf{p}^{(j)}), F(\mathbf{p}^{(j)}))$.
(b) $\mathbf{x}^{(j)}$ agrees with the observations $\mathbf{z}^{(j)}$.

Method:

The idea is to find $F(\cdot, \mathbf{p})$ and $\mathbf{x}^{(j)}$ that minimize both

(a) sum of sub-optimality gaps $r_{\mathbf{eq}}$ of the VI's
(b) observation residual $r_{\mathbf{obs}} := \sum_{j=1}^{N} \phi\left(g(\mathbf{x}^{(j)}, \mathbf{p}^{(j)}),\ \mathbf{z}^{(j)}\right)$

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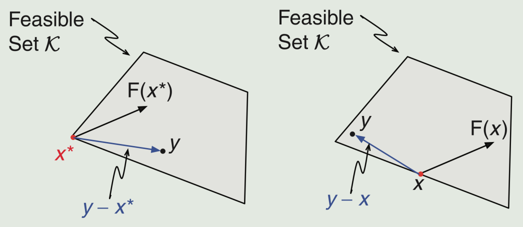
Formalization

Define the optimality gap for VI($\mathcal{K}, F$):

$$r_{\mathrm{VI}}(\mathbf{x}) = \max_{\text{by} \in \mathcal{K}} F(\mathbf{x})^T(\mathbf{x} - \text{by}) \tag{4}$$

Note that

- $r_{\mathrm{VI}}(\mathbf{x}) \geq 0, \ \forall \, \mathbf{x} \in \mathcal{K}$
- $r_{\mathrm{VI}}(\mathbf{x}) \leq \epsilon \quad \Longleftrightarrow \quad \min_{\text{by} \in \mathcal{K}} F(\mathbf{x})^T(\text{by} - \mathbf{x}) \geq -\epsilon$



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Residual of the primal-dual system

Define residual function for $\mathbf{x} \in \mathcal{K}$ and by $\in \mathbb{R}^n$

$$r_{\text{PD}}(\mathbf{x}, \text{ by }) = F(\mathbf{x})^T \mathbf{x} - \mathbf{b}^T \text{ by} \tag{5}$$

Theorem [1]

The following holds for any $\epsilon \geq 0$ and $\mathbf{x} \in \mathcal{K}$

$$r_{\text{VI}}(\mathbf{x}) \leq \epsilon \iff \exists \text{ by } \in \mathbb{R}^n : \mathbf{A}^T \text{ by } \leq F(\mathbf{x}),\ r_{\text{PD}}(\mathbf{x}, \text{ by }) \leq \epsilon \tag{6}$$

Set $r_{\text{eq}} = \sum_j r_{\text{PD}}(\mathbf{x}^{(j)}, \text{ by }^{(j)}, \mathbf{p}^{(j)}) = \sum_j F(\mathbf{x}^{(j)}, \mathbf{p}^{(j)})^T \mathbf{x}^{(j)} - \mathbf{b}^T(\mathbf{p}^{(j)}) \text{ by }^{(j)}$ and solve:

$$\begin{aligned} \min_{F, \mathbf{x}, \text{ by}} \quad & w_{\text{eq}}\, r_{\text{eq}} + w_{\text{obs}}\, r_{\text{obs}} \\ \text{s.t.} \quad & \mathbf{x}^{(j)} \in \mathcal{K}(\mathbf{p}^{(j)}), \quad \mathbf{A}(\mathbf{p}^{(j)})^T \text{ by }^{(j)} \leq F(\mathbf{x}^{(j)}, \mathbf{p}^{(j)}), \quad \forall j \end{aligned} \tag{7}$$

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[1]Dimitris Bertsimas, Vishal Gupta, and Ioannis Paschalidis. Data-driven estimation in equilibrium using inverse optimization. *Math. Program.*, pages 595–633, 2015

Previous works

Inverse Variational Inequality formulation :

$$\min\ r_{\text{eq}} \quad \text{s.t.} \quad r_{\text{obs}} = 0,\ \text{primal and dual feasibility} \tag{8}$$

- assumes complete and noiseless observations
- not robust to measurement errors

Bilevel programming formulation:

$$\min\ r_{\text{obs}} \quad \text{s.t.} \quad r_{\text{eq}} = 0 \tag{9}$$

- difficult to solve due to bilevel structure
- KKT system as constraints pose numerical difficulties (ref)

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[1]Dimitris Bertsimas, Vishal Gupta, and Ioannis Paschalidis. Data-driven estimation in equilibrium using inverse optimization.
Math. Program., pages 595–633, 2015

Pareto optimization

Solve the Weighted Sum Program (WSP) :

$$\min\ w_{eq} r_{eq} + w_{obs} r_{obs} \quad \text{s.t.} \quad \text{primal and dual feasible} \tag{10}$$

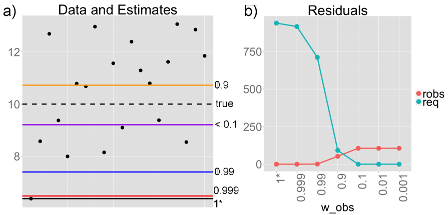
Algorithm 1 `Weighted-sum(·)` Weighted sum method

1: Normalize objectives r_{eq} and r_{obs} for consistent comparisons.

2: Solve with $w_{\text{obs}} + w_{\text{eq}} = 1$ and $w_{\text{obs}} \in \{0.001, 0.01, 0.1, 0.9, 0.99, 0.999\}$

3: Check the values of r_{eq} and r_{obs}.

- Robust to noise or outliers with appropriate choice of ϕ in r_{obs}
- smoothing with penalization r_{PD} instead of constraint $r_{PD} = 0$



[7]R.T. Marler and J.S. Arora. Survey of multi-objective optimization methods for engineering. *Struct. Multidiscip. Optim.*, page 369âĂŞ395, 2004

Asymptotic behavior

Let $r_{\text{eq}}^\star$ be the optimal obj. value of the inverse VI $\{\min\ r_{\text{eq}} \text{ s.t. } r_{\text{obs}} = 0\}$

Let $r_{\text{obs}}^\star$ be the optimal obj. value of the BP $\{\min\ r_{\text{obs}} \text{ s.t. } r_{\text{eq}} = 0\}$

Let the WSP $\{\min\ w_{\text{eq}} r_{\text{eq}} + w_{\text{obs}} r_{\text{obs}} \text{ s.t. feasibility}\}$ with $w_{\text{eq}} + w_{\text{obs}} = 1$.

Theorem [9]

Any optimal solution $u^\star \in \mathcal{S}(w_{\text{eq}}, w_{\text{obs}})$ to the WSP is such that

$$r_{\text{obs}}(\mathbf{u}^\star) \le r_{\text{eq}}^\star (w_{\text{obs}}^{-1} - 1) \approx r_{\text{eq}}^\star (1 - w_{\text{obs}}) \text{ as } w_{\text{obs}} \xrightarrow{\le} 1 \tag{11}$$

$$r_{\text{eq}}(\mathbf{u}^\star) \xrightarrow{\le} r_{\text{eq}}^\star \quad \text{(uniformly)} \tag{12}$$

Given compactness and any sequence $w_{\text{obs}}^{(n)} \longrightarrow 1$, there exists a sequence $\{\mathbf{u}^{(n)}\}_n \in \{\mathcal{S}(w_{\text{eq}}^{(n)}, w_{\text{obs}}^{(n)})\}_n$ and a sub-sequence of it converging to a solution to the inverse VI.

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[9]Jerome Thai and Alexandre Bayen. Imputing a variational inequality or convex objective function: a robust approach.
Journal of Mathematical Analysis and Applications, 2016

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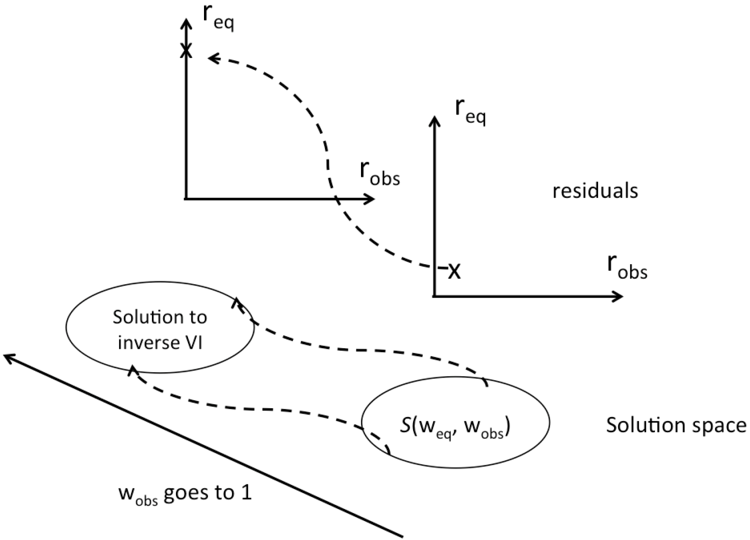
Asymptotic behavior

Opens a new dimension for which the 'edge' are the inverse VI and the BP.



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Thank you!

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