Effect of Information in Bayesian Congestion Games

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Motivation

- Inherent heterogeneities in both access and accuracy of TIS
- Asymmetrically informed travelers choose routes based on private beliefs
- Our work is related to well-known literature on congestion game and value of information.
- Our work is also related to the recent papers: Acemoglu et al. (2016), Krichene et al. (2016), Ratliff et al. (2016)

Outline

Information penetration effects qualitative properties of equilibrium outcome.

Bayesian Congestion Games — Information Heterogeneity

Equilibrium Characterization — Change sizes of asymmetrically informed population

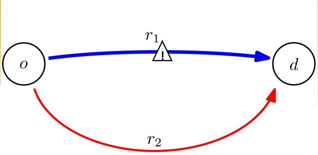
Equilibrium Cost Analysis — Complete order of population costs
Bounds on average cost

Motivating Example

$$c_1^s(f_{r_1}) = \begin{cases} \alpha_1^{\mathbf{a}} f_{r_1} + b, & s = \mathbf{a}, \\ \alpha_1^{\mathbf{n}} f_{r_1} + b, & s = \mathbf{n}. \end{cases}$$

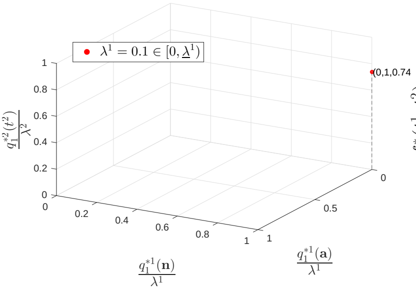
$$c_2(f_{r_2}) = \alpha_2 f_{r_2} + b,$$

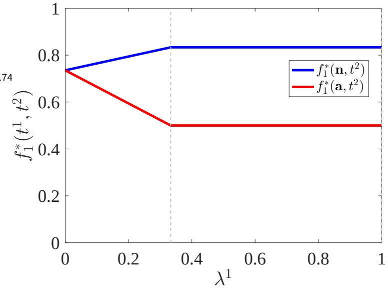
$$\alpha_1^{\mathbf{n}} < \alpha_2 < \alpha_1^{\mathbf{a}}, \quad D = 1.$$



- Population 1 is completely informed, population 2 has no information

Figure: Effect of information penetration





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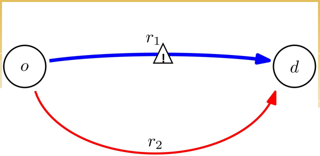

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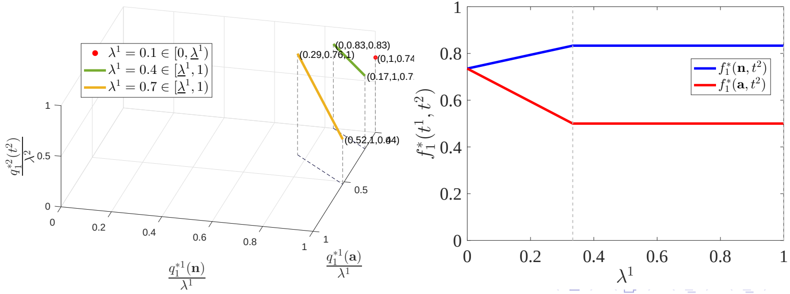

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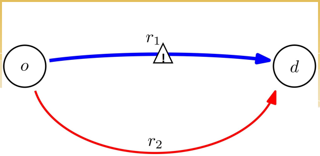

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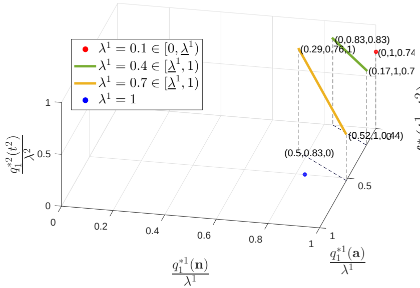
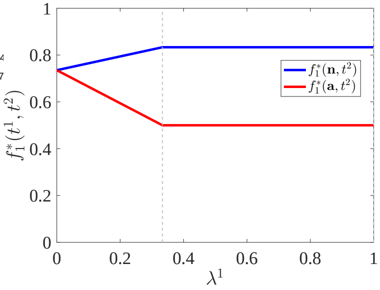
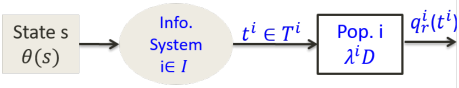


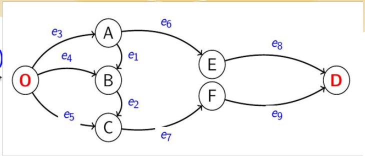


Model

Single o-d pair

Private knowledge:
Type t^1 , t^2





Common knowledge:
Common prior π(s, t)
Population size $\lambda = \left(\lambda^i\right)_{i\in I} \in \Delta(I)$
Total demand D

Route flow: $f_r(t) = \sum_{i\in I} q^i_r(t^i)$
Edge load: $w_e(t) = \sum_{r\ni e} f_r(t)$
Costs: $c^s_e\left(w_{e(t)}\right)$ is increasing with $w_{e(t)}$

- $\mu^i(s, t^{-i}|t^i)$: interim belief of type t^i obtained from common prior:

$$\mu^i(s, t^{-i}|t^i) = \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)}, \quad \forall s \in \mathcal{S}, \quad \forall t^{-i} \in \mathcal{T}^{-i}$$

- Expected cost of route $r \in \mathcal{R}$ based on the interim belief $\mu^i(s, t^{-i}|t^i)$:

$$\mathbb{E}[c_r(q)|t^i] = \sum_{s\in\mathcal{S}} \sum_{t^{-i}\in\mathcal{T}^{-i}} \sum_{e\in r} \mu^i(s, t^{-i}|t^i) c^s_e(w_e(t^i, t^{-i})), \quad \forall t^i \in \mathcal{T}^i, \forall i \in \mathcal{I}, \forall r \in \mathcal{R}.$$

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Model

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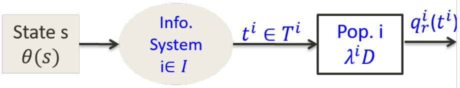

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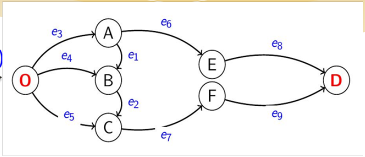

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Bayesian Wardrop Equilibrium

- Feasible strategy set $\mathcal{Q}(\lambda)$:

$$\sum_{r \in \mathcal{R}} q_r^i(t^i) = \lambda^i \mathrm{D}, \quad \forall i \in \mathcal{I}, t^i \in \mathcal{T}^i,$$

$$q_r^i(t^i) \geq 0, \quad \forall r \in \mathcal{R}, i \in \mathcal{I}, t^i \in \mathcal{T}^i.$$

- A strategy profile $q^* \in \mathcal{Q}(\lambda)$ is a Bayesian Wardop Equilibrium (BWE) if for any $i \in \mathcal{I}$ and $t^i \in \mathcal{T}^i$:

$$\forall r \in \mathcal{R}, q_r^{*i}(t^i) > 0 \quad \Rightarrow \quad \mathbb{E}[c_r(q^*)|t^i] \leq \mathbb{E}[c_{r'}(q^*)|t^i], \quad \forall r' \in \mathcal{R}.$$

- Equilibrium population cost: expected cost of travelers in one population

$$C^{*i}(\lambda) \triangleq \frac{1}{\lambda^i \mathrm{D}} \sum_{t^i \in \mathcal{T}^i} \Pr(t^i) \sum_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i] q_r^{*i}(t^i).$$

- Equilibrium average cost: expected cost of travelers in all populations

$$C^*(\lambda) = \sum_{i \in \mathcal{I}} \lambda^i C^{*i}(\lambda).$$

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Weighted Potential Game

- Game Γ is a weighted potential game:

$$\Phi\left(q\right) \triangleq \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi\left(s,t\right)\int_0^{\sum_{r\ni e}\sum_{i\in\mathcal{I}} q_r^i(t^i)} c_e^s(z)dz,$$

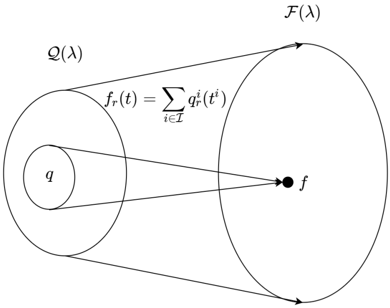
  where $\gamma(t^i) = \Pr(t^i)$.
- A strategy profile $q \in \mathcal{Q}^*(\lambda)$ if and only if it is an optimal solution of the following convex optimization problem:

$$\begin{aligned} \min \quad & \Phi(q) \\ s.t. \quad & q \in \mathcal{Q}(\lambda), \end{aligned} \qquad \text{(OPT-}\mathcal{Q}\text{)}$$

- The equilibrium edge load $w^*(\lambda)$ is unique.

$\mathcal{Q}(\lambda) \to \mathcal{F}(\lambda)$

It is hard to characterize how $\mathcal{Q}^*(\lambda)$ changes with λ directly from (OPT-$\mathcal{Q}$).
Key step: feasible strategy set $\mathcal{Q}(\lambda) \longrightarrow \mathcal{F}(\lambda)$ feasible route flow set:



We change (OPT-$\mathcal{Q}$) to (OPT-$\mathcal{F}$).
Equilibrium route flows can be computed as optimal solutions in (OPT-$\mathcal{F}$).

(OPT-$\mathcal{F}$)

- Step 1: Feasible route flow set $\mathcal{F}(\lambda)$:

$$f_r(t^i, t^{-i}) - f_r(\tilde{t}^i, t^{-i}) = f_r(t^i, \tilde{t}^{-i}) - f_r(\tilde{t}^i, \tilde{t}^{-i}), \quad \forall r \in \mathcal{R}, \forall t^i, \tilde{t}^i \in \mathcal{T}^i, \forall t^{-i}, \tilde{t}^{-i} \in \mathcal{T}^{-i}, \forall i \in \mathcal{I}, \tag{1}$$

$$\sum_{r \in \mathcal{R}} f_r(t) = D, \quad \forall t \in \mathcal{T}, \tag{2}$$

$$f_r(t) \geq 0, \quad \forall r \in \mathcal{R}, \quad \forall t \in \mathcal{T}, \tag{3}$$

$$\mathrm{D} - \sum_{r \in \mathcal{R}} \min_{t^i \in \mathcal{T}^i} f_r(t^i, t^{-i}) \leq \lambda^i \mathrm{D}, \quad \forall t^{-i} \in \mathcal{T}^{-i}, \quad \forall i \in \mathcal{I}, \tag{4}$$

(OPT-$\mathcal{F}$)

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Constraints (1)-(3) are independent of λ

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Impact of Information:

$$J^i(f) = \mathrm{D} - \sum_{r \in \mathcal{R}} \min_{t^i \in \mathcal{T}^i} f_r(t^i, t^{-i}) = \sum_{r \in \mathcal{R}} \max_{t^i \in \mathcal{T}^i} \left(q_r^i(\widehat{t}^i) - q_r^i(t^i) \right)$$

Impact of information is the summation over all r of the maximum change in population i's demand on r when the signal changes from $\widehat{t}^i$ to any other types

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(4-i): λ^i effects the equilibrium by bounding the impact of information on population i

(OPT-$\mathcal{F}$)

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- Step 2: A feasible route flow $f \in \mathcal{F}^*(\lambda)$ if and only if it is optimal solution of:

$$\begin{aligned} \min \quad & \widehat{\Phi}(f) = \sum_{s\in\mathcal{S}} \sum_{e\in\mathcal{E}} \sum_{t\in\mathcal{T}} \pi(s,t) \int_0^{\sum_{r\ni e} f_r(t)} c_e^s(z)dz, \\ s.t. \quad & (1)\text{-}(4). \end{aligned} \tag{OPT-$\mathcal{F}$}$$

  - (OPT-$\mathcal{F}$) is a convex optimization program

Equilibrium Characterization: Change λ Directionally

Consider any two populations i and j, and the size vector of the remaining populations $\lambda^{-ij} = (\lambda^k)_{k \in \mathcal{I} \setminus \{i,j\}}$. Total size $|\lambda^{-ij}| = \sum_{k \in \mathcal{I} \setminus \{i,j\}} \lambda^k$.

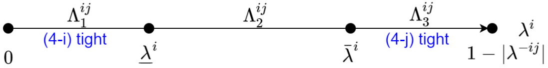


Figure: Change relative size between population i and j

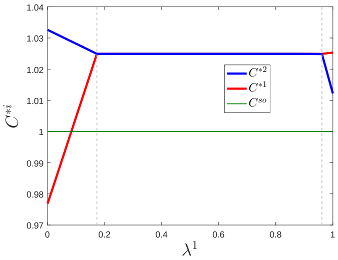
- $0 \leq \underline{\lambda}^i \leq \bar{\lambda}^i \leq 1 - |\lambda^{-ij}|$, both can be computed from convex optimization in polynomial time
- The impact of information on the minor population is limited by its size:
  In regime Λ_1^{ij}, population i is minor, $J^i(f^*) = \lambda^i \mathrm{D}$
  In regime Λ_3^{ij}, population j is minor, $J^j(f^*) = \lambda^j \mathrm{D}$
- In the middle regime Λ_2^{ij}, the impact of information is not limited by the size of either population

Equilibrium Characterization: Change λ Directionally

$\Pr(t^1 = s|s) = 0.8$, $\Pr(t^2 = s|s) = 0.6$

Two side regimes: Λ_1^{ij}, Λ_3^{ij}

- The minor population has lower cost:
  In regime Λ_1^{ij}, $C^{*i}(\lambda) < C^{*j}(\lambda)$
  In regime Λ_3^{ij}, $C^{*i}(\lambda) > C^{*j}(\lambda)$
- Equilibrium edge load $w^*(\lambda)$ changes with λ

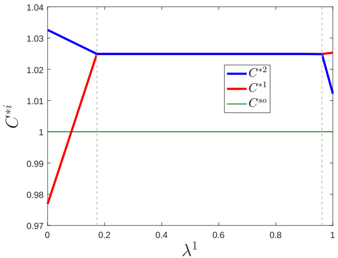


Equilibrium Characterization: Change λ Directionally

$\Pr(t^1 = s|s) = 0.8$, $\Pr(t^2 = s|s) = 0.6$

Middle regime: Λ_2^{ij}

- Both populations have identical costs in equilibrium
- Equilibrium edge load $w^*(\lambda)$ does not change with λ



Effect of Information Penetration

- The equilibrium edge load does not change with the size of one population if and only if the impact of information on that population is not limited by its size
- The thresholds $\underline{\lambda}^i$, $\bar{\lambda}^i$ are dependent on λ^{-ij} and π, but the qualitative properties of equilibrium in each regime is robust with λ^{-ij} and π
- Since any feasible change of λ can be written as a linear combination of directional change, how equilibrium property changes in an arbitrary direction can be obtained by analyzing the regimes in each direction.

Equilibrium Characterization: Intermediate Set $\Lambda^\dagger$

For any given π, there exists a non-empty intermediate set $\Lambda^\dagger$, where the equilibrium outcome is independent with λ.

- All populations have the same costs in equilibrium
- The impact of information is not limited by the size of any population
- The equilibrium edge load $w^*(\lambda)$ does not change with λ

Order of Equilibrium Population Costs

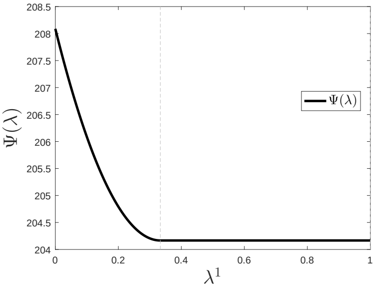
A complete order of $C^{*i}(\lambda)$ can be obtained in polynomial time

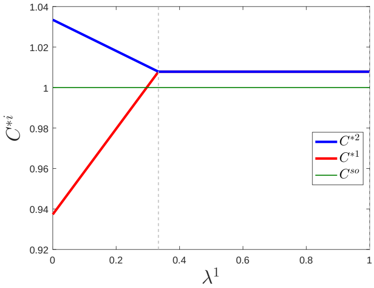
- No assumption on π. Especially, we do not specify any one information system is sufficient than another.
- Since when comparing any pair of populations, π effects the thresholds, the order is dependent on π
- We sort all $C^{*i}(\lambda)$ by comparing pairs of populations for no more than $|\mathcal{I}|^2$ times, and the order is dependent on λ

One Population Uninformed

If population j is uninformd, $C^{*i}(\lambda) \leq C^{*j}(\lambda)$ for any $i \in \mathcal{I}$.

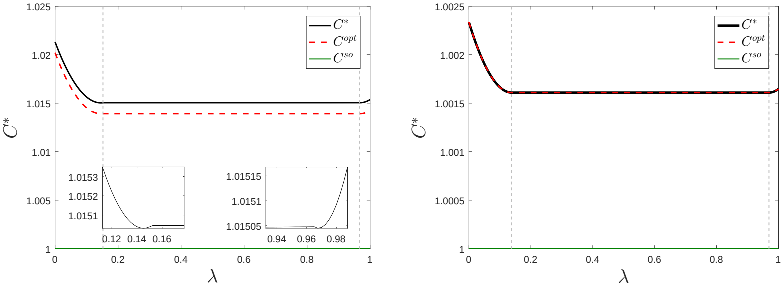
- $\bar{\lambda}^i = 1 - |\lambda^{-ij}|$. Three regimes reduce to two.
- The uninformed population always has the highest cost





Equilibrium Average Cost

- We provide tight lower bound of equilibrium average cost $C^*(\lambda)$.
- Under certain conditions, $C^*(\lambda)$ is minimized by any $\lambda \in \Lambda^\dagger$.
- The bound on worst case inefficiency in equilibrium is identical to that in complete information game



Conclusion

Information penetration λ effects the impact of information in equilibrium, order of equilibrium population costs and equilibrium average cost

- The population size vector effects the equilibrium outcome by limiting the impact of information on each population. The equilibrium outcome does not change with population sizes if and only if the impact of information is fully achieved.
- A complete order of population equilibrium costs can be obtained in polynomial time given any common prior and population size vector
- We analyze lower bound on equilibrium average cost, minimum equilibrium average cost, and the worst case inefficiency