

H_∞ -Optimal Control of MIMO Networks with Delays

We use Full-State Feedback

Form of Multi-Delay System:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_i A_i x(t - \tau_i) + B_1 w(t) + B_2 u(t), \\ y(t) &= Cx(t) + D_1 w(t) + D_2 u(t)\end{aligned}$$

Full-State Feedback: 3 Gains - K_0 , K_{1i} , $K_{2i}(s)$

$$u(t) = K_0 x(t) + \sum_i K_{1i} x(t - \tau_i) + \sum_i \int_{-\tau_i}^0 K_{2i}(s) x(t + s) ds$$

Requires Measurement of History

A Scalable Algorithm

No Approximation! No Conservatism!

We can handle 50+ states and delays

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.438	.172	.266	1.24	17.2
2	.269	.643	2.932	17.1	647.2
3	.627	2.634	10.736	91.43	5170.2
5	1.294	13.12	84.77.7	1877	65281
10	11.41	469.86	4439	57894	NA

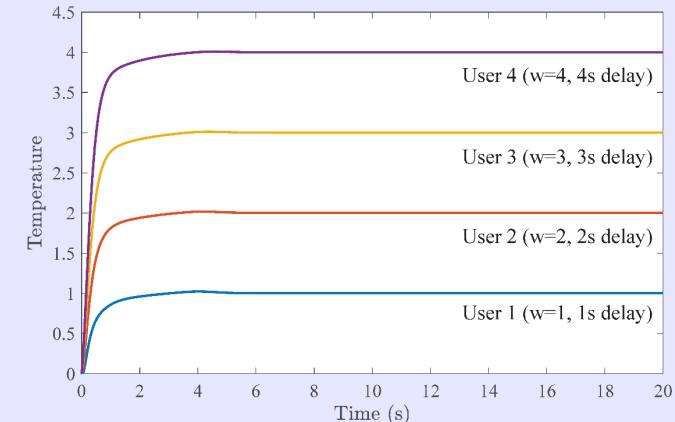
Table: CPU sec indexed by # of states (n) and # of delays (K)

Network of Showering Guests:

- T_{1i} is tap position
- T_{2i} is water temperature

$$\dot{T}_{1i}(t) = T_{2i}(t) - w_i(t)$$

$$\begin{aligned}\dot{T}_{2i}(t) &= -\alpha_i (T_{2i}(t - \tau_i) - w_i(t)) \\ &+ \sum_{j \neq i} \gamma_{ij} \alpha_j (T_j(t - \tau_j) - w_j(t)) + u_i(t)\end{aligned}$$



Illustrated: 8 states, 4 delays, 12 outputs

Generalizes the Classic LMI:

Min γ s.t. $\exists P > 0$ and Z s.t.

$$\begin{bmatrix} PA^T + AP + Z^T B_2^T + B_2 Z & *^T & *^T \\ B_1^T & -\gamma I & *^T \\ C_1 P + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \leq 0$$

P and Z become operators, parameterized by matrices.

H_∞ -Optimal Estimation of MIMO Networks with Delays

Estimate the Full-State!

Form of Multi-Delay System:

$$\dot{x}(t) = A_0x(t) + \sum_i A_i x(t - \tau_i) + Bw(t),$$

$$z(t) = C_{10}x(t) + \sum_i C_{1i}x(t - \tau_i),$$

$$y(t) = C_2x(t).$$

We can handle 50+ states and delays

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.516	.218	.375	2.203	24.094
2	.219	.547	2.141	19.282	875.137
3	.3910	1.782	9.484	113.236	4742.7
5	1.375	12.454	109.939	1859.9	62069
10	18.406	582.945	4717.2	66033	N/A

Table: CPU sec for n states and K delays

A Scalable Algorithm

No Approximation! No Conservatism!

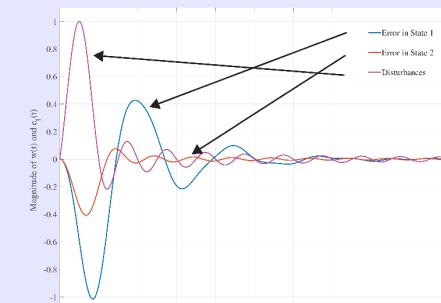
Generalizes the LMI:

Min γ s.t. $\exists P > 0$ and Z s.t.

$$\begin{bmatrix} PA + ZC_2 + (PA + ZC_2)^T & *^T & *^T \\ -(PB + ZD)^T & -\gamma I & *^T \\ C_{10} & 0 & -\gamma I \end{bmatrix} \leq 0$$

P and Z become operators, parameterized by matrices.

Example of Error Dynamics:



A PDE Observer

(observed errors)(nominal dynamics)(corrective gains)

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + \sum_i A_i \hat{\phi}(t, -\tau_i) + \mathcal{L}_1 e_0(t) + \sum_i \mathcal{L}_{2i} e_i(t - \tau_i) + \sum_i \int_{-\tau_i}^0 \mathcal{L}_{3i}(\theta) e_i(t + s) d\theta,$$

$$\partial_t \hat{\phi}_i(t, s) = \partial_s \hat{\phi}_i(t, s) + \mathcal{L}_{4i}(s) e_0(t) + \sum_j \mathcal{L}_{5ij}(s) e_j(t - \tau_j) + \mathcal{L}_{6i}(s) e_i(t + s) + \sum_j \int_{-\tau_i}^0 \mathcal{L}_{7ij}(s, \theta) e_j(t + \theta) d\theta,$$

$$\hat{\phi}_i(t, 0) = \hat{x}(t),$$

$$e_0(t) = C_2\hat{x}(t) - y(t), \quad e_i(t + s) = C_2\phi_i(t, s) - y(t + s),$$

$$z_e(t) = C_{10}e_0(t) + \sum_i C_{1i}e_i(t - \tau_i)$$

Reconstructs Past and Current States.

Looking Forward: Lots to Do

For the Next Year

- Input Delays
- Sampled-Data
- Optimal Estimator-Based Controllers
- Parametric Uncertainty
- Time-Varying Delay

Extension to Coupled PDEs (CDC 2018)

A universal formulation

$$\mathbf{x}_t(s, t) = A_0(s)\mathbf{x}(s, t) + A_1(s)\mathbf{x}_s(s, t) + A_2(s)\mathbf{x}_{ss}(s, t)$$

where $\mathbf{x} : [a, b] \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$

with Generalized BCs:

$$B \begin{bmatrix} x(a, t) & x(b, t) & x_s(a, t) & x_s(b, t) \end{bmatrix}^T = 0$$

Where B is rank 2n

Accuracy Rigorously Verified

- Compared to 10th-order Padé Approximation

Efficient Implementation

- Controller Reconstruction
- Optimized for Simulation
- Available on CodeOcean

Generalize more LMIs:

Many more LMIs can be efficiently adapted to delayed networks

H_∞ -Gain Analysis:

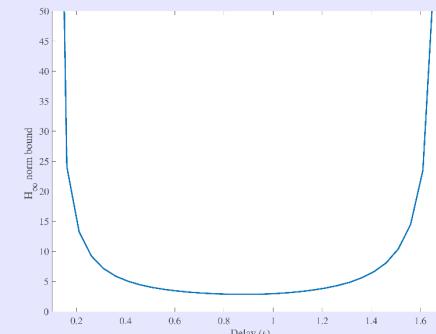


Figure: Bound on H_∞ norm of a multi-delay system vs. delay

2-Part presentation at ACC 2019:

- In Special Session on ∞ -D systems