CONTROL AND STABILITY RESULTS FOR SYMMETRIC AND *APPROXIMATELY SYMMETRIC* NONLINEAR CYBER-PHYSICAL SYSTEMS

NSF CPS Science of System Integration Review Meeting

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April 23, 2013

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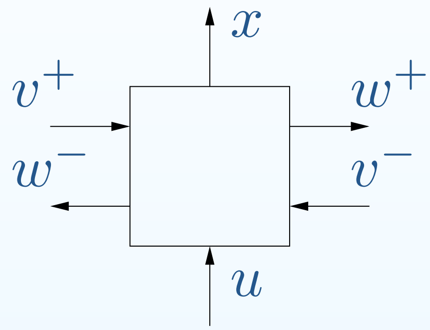
   (d) Nonautonomous boundedness about set: in progress (ICRA 2013?).

Symmetric Systems have Efficient Tests for Stability

- Overall goal: results for system integration through compositionality.
- Compositionality: system-level properties can be computed from local properties of components
  - System-level properties *preserved* when expanding system.
- Initial focus: invariant properties for symmetric systems.
- Emphasis on general results, not limited to specific system dynamics.
- Initial results: symmetric systems and stability.
- These results are Lyapunov-based, so the natural extension is to passivity.
- Also working toward use of *approximate* symmetries.
- Prior and related work: [5, 8, 6, 7, 10, 9, 1, 12]

Symmetric Systems can be Constructed from Components

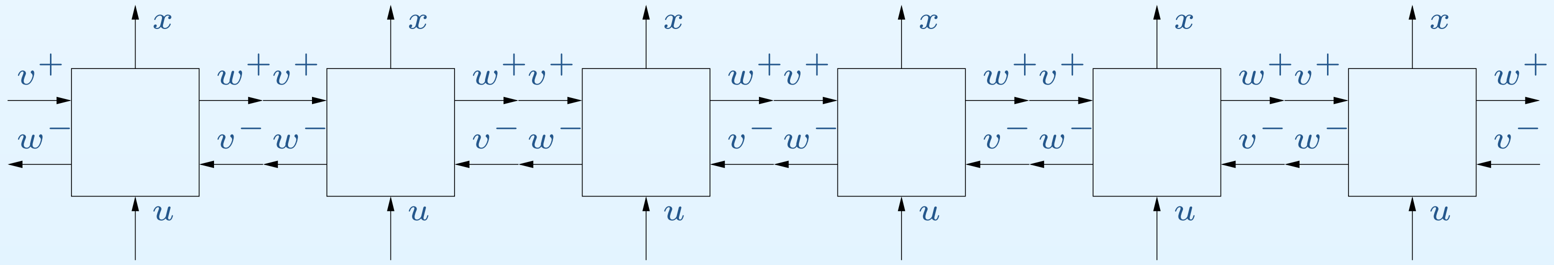
Consider a *basic building block element*:



where

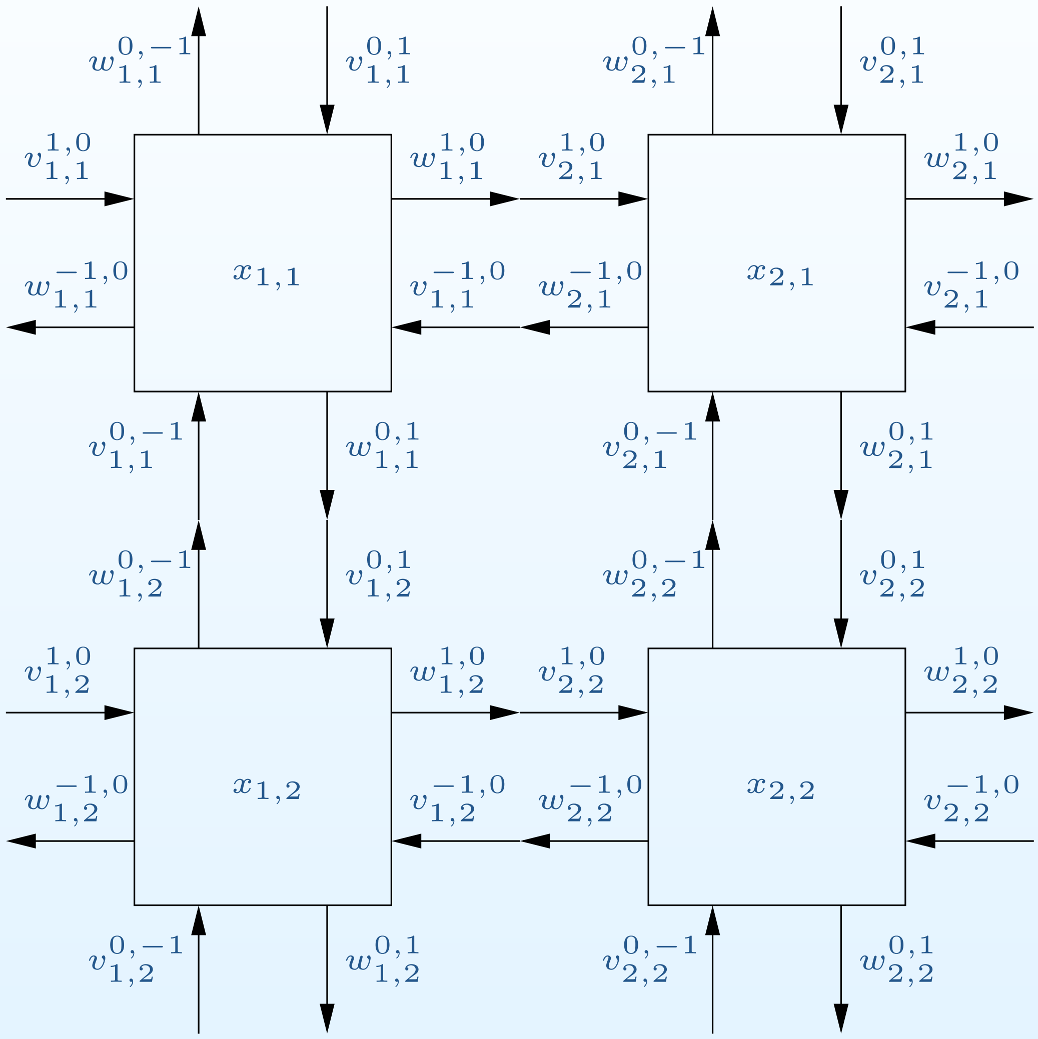
- x is the state vector;
- u is the vector of control inputs;
- $w^{\pm}$ are the *outputs*; and,
- $v^{\pm}$ are the coupling inputs.

Connecting the inputs to the outputs gives



Periodic Interconnections are One Aspect of a Symmetric System

Systems can be built with various topologies:



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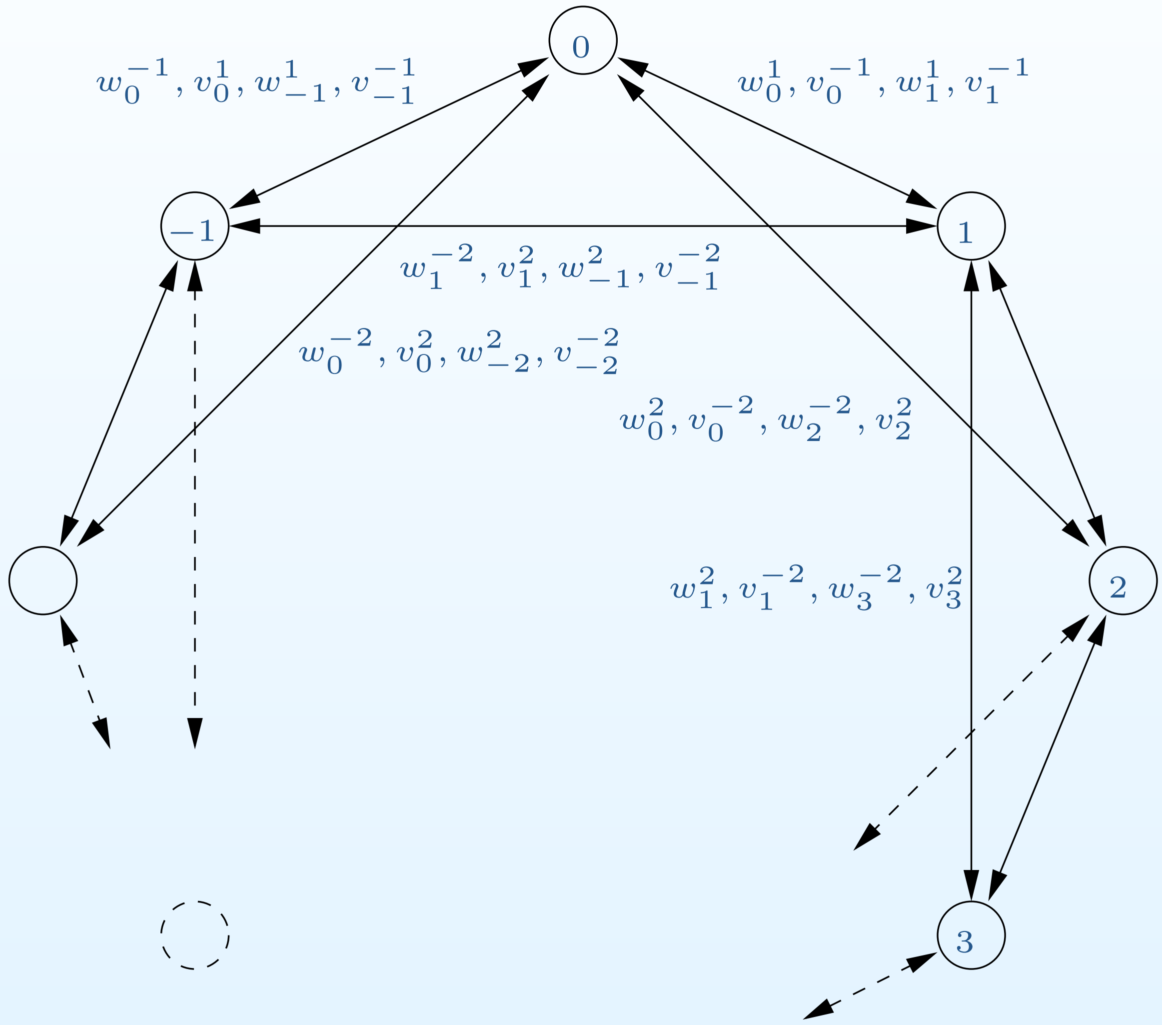
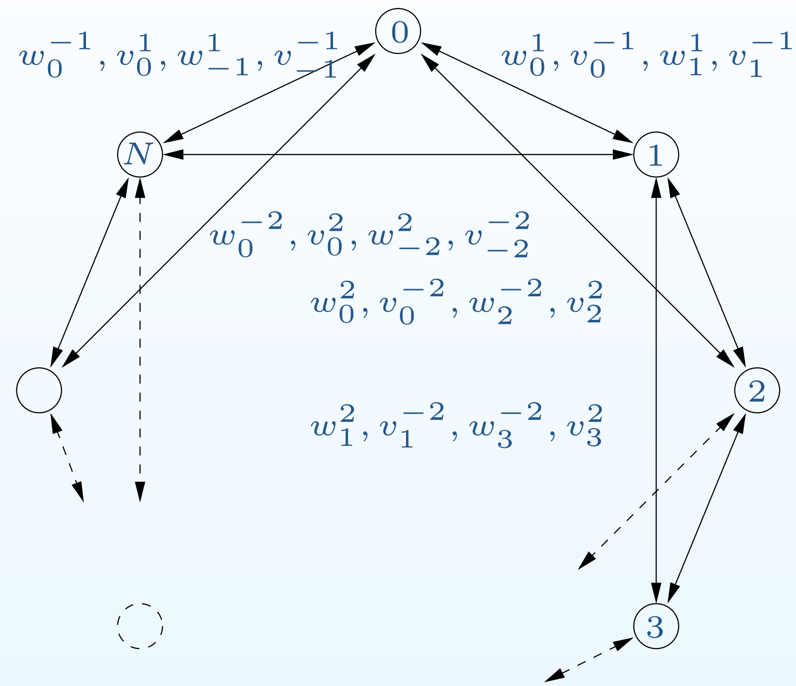


Group Theory Allows for Generalization to Complicated Interconnection Topologies

- Represent connectedness with *generators*, $\{s_1, \ldots, s_n\}$ with $g_2 = s_i g_1$.
- *Cayley Graph*:
  - nodes = components
  - edges = communication
- *Equivalent connections* for two systems if they have the same generators.
- Example: $S = \{-2, -1, 1, 2\}$.
- Each component:

$$\dot{x}_i = f_i(x) + g_i(x)u$$
$$w_i^s(t) = w_i^s(x_i(t)).$$



- Periodic interconnections:

$$v_g^s(t) = w_{s^{-1}g}^s\left(x_{s^{-1}g}(t)\right)$$
$$w_g^s(t) = v_{sg}^s\left(x_g(t)\right)$$

A Symmetric System = Periodic Interconnections + Identical Component Dynamics

A *symmetric system* has components with identical dynamics: $f_{g_1}(x) = f_{g_2}(x)$, $g_{g_1,j}(x) = g_{g_2,j}(x)$, $w^{s}_{s^{-1}g_1}(x) = w^{s}_{s^{-1}g_2}(x)$ and identical control laws

$$u_{g_1,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_1}(x_2(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_1}(x_{|X|+1}(t))\right) = u_{g_2,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_2}(x_2(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_2}(x_{|X|+1}(t))\right)$$

for all $g_1 \in G_1$, $g_2 \in G_2$, $s \in X$, $x \in \mathbb{R}^n$, $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$ where $m = m_{g_1} = m_{g_2}$.

An Equivalence Class of Symmetric Systems Formalizes the Idea of “Adding or Removing” Components

Two systems are *equivalent* if they are both symmetric with equal components and the *same generators*.

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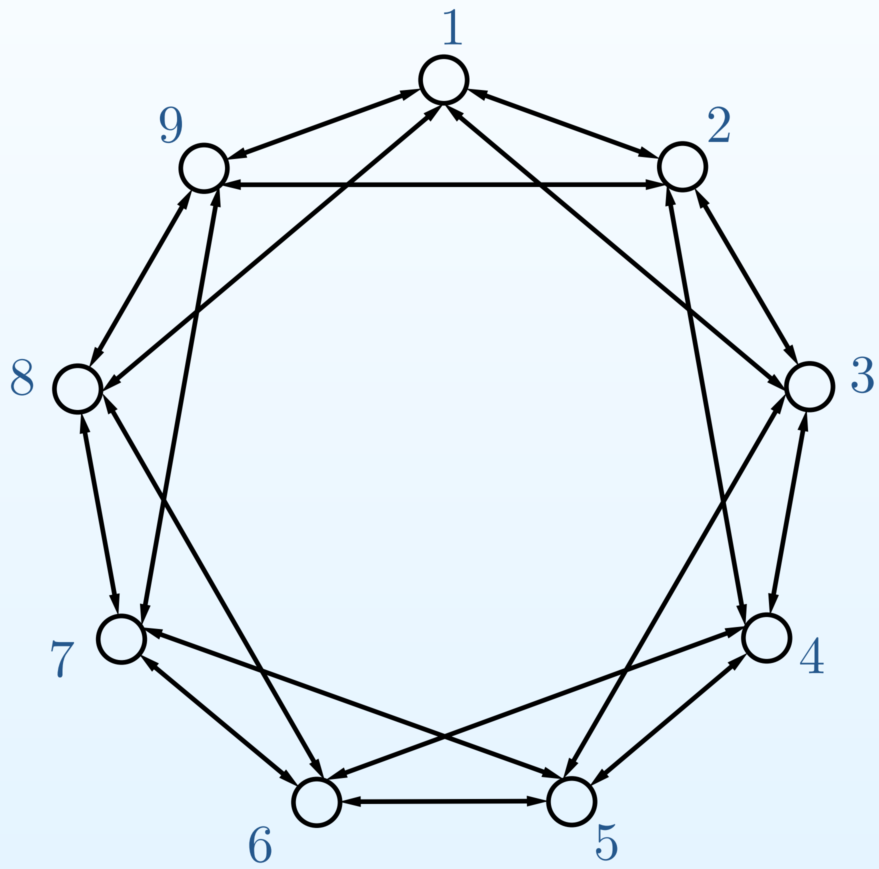

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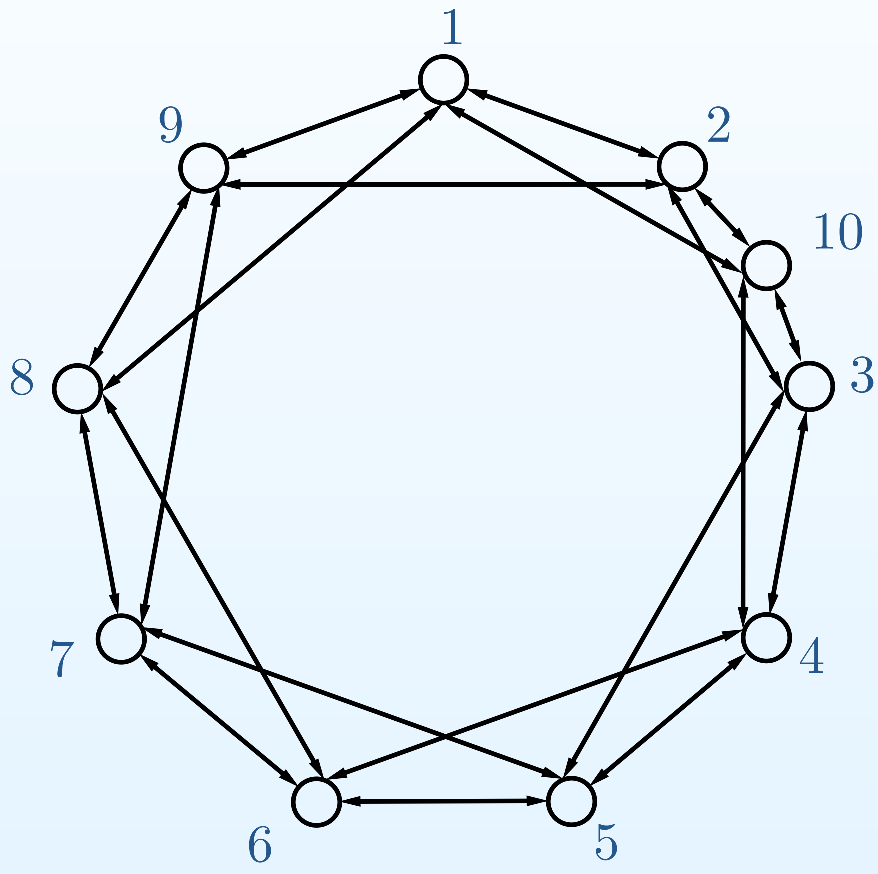

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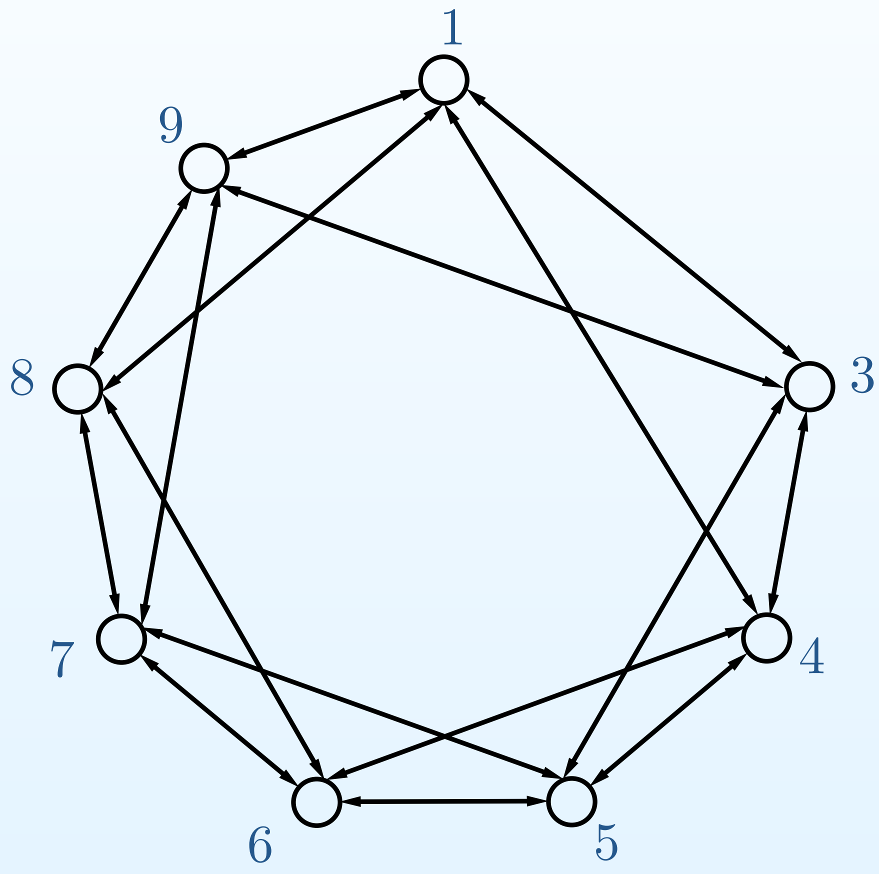

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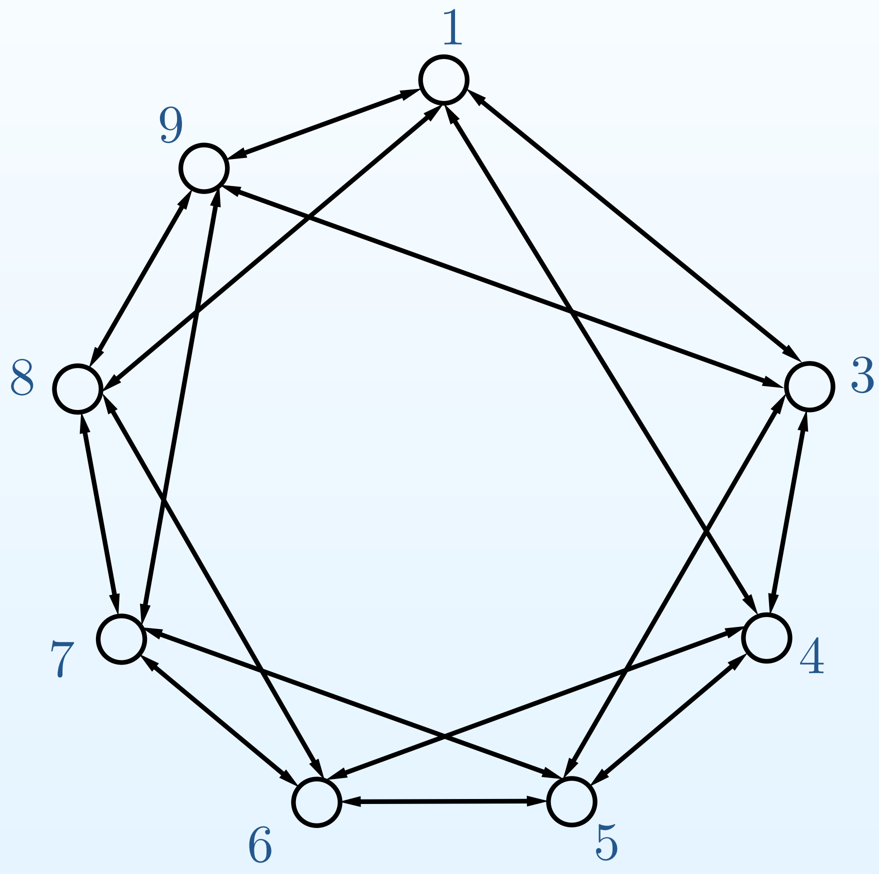

What properties are invariant throughout the *entire equivalence class*?

Conditions on V Provide Stability for Entire Equivalence Class

- **Stability:** if a symmetric distributed system is stable, so is any equivalent symmetric system, so can "grow" or "shrink."
  - Growing: useful for analysis/design on a small system with guaranteed invariance for larger equivalent systems
  - Shrinking: reconfigurable robustness
- **Robustness:** stability in the sense of Lyapunov is guaranteed even when components fail without any reconfiguration necessary.
- Requires a symmetric Lyapunov function: $V = \sum_{i \in G} V_i$ where

$$V_{g_1}\left(x_1, w^{s_1}_{s_1^{-1}g_1}(x_2), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_1}(x_{|X|+1})\right) = V_{g_2}\left(x_1, w^{s_1}_{s_1^{-1}g_2}(x_2), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_2}(x_{|X|+1})\right)$$

for all $g_1, g_2 \in G$ and $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

Formation Control Example Demonstrates Stability and Robustness

- From [11] second-order mechanical system agents:

$$\frac{d}{dt}\begin{bmatrix} x_i \\ \dot{x}_i \\ y_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_i \\ 0 \\ \dot{y}_i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{i,1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{i,2}. \quad (1)$$

- Goal= regular $(N+1)$-polygon centered at the origin, hence

$$d_{ij} = \begin{cases} 1, & |i-j| = 1 \\ \frac{\sin\left(\frac{2\pi}{N+1}\right)}{\sin\left(\frac{\pi}{N+1}\right)}, & |i-j| = 2 \end{cases} \quad \text{and} \quad r_i = \frac{1}{2\sin\frac{\pi}{N}}.$$

- Take the control law to be $u =$

$$-\sum_j \begin{bmatrix} \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(x_i-x_j) \\ \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(y_i-y_j) \end{bmatrix} - k_d \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} - \begin{bmatrix} \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i \\ \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} y_i \end{bmatrix}.$$

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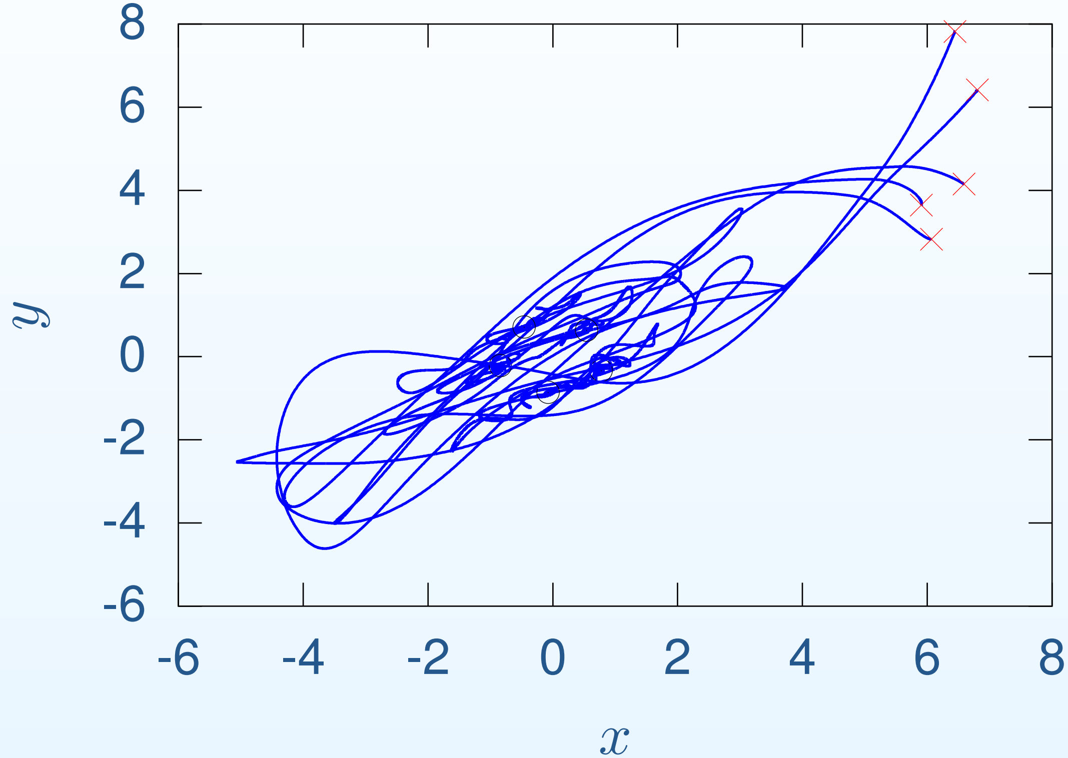

Figure 1. Five robot stable formation control.

Formation Control Example Demonstrates Stability and Robustness

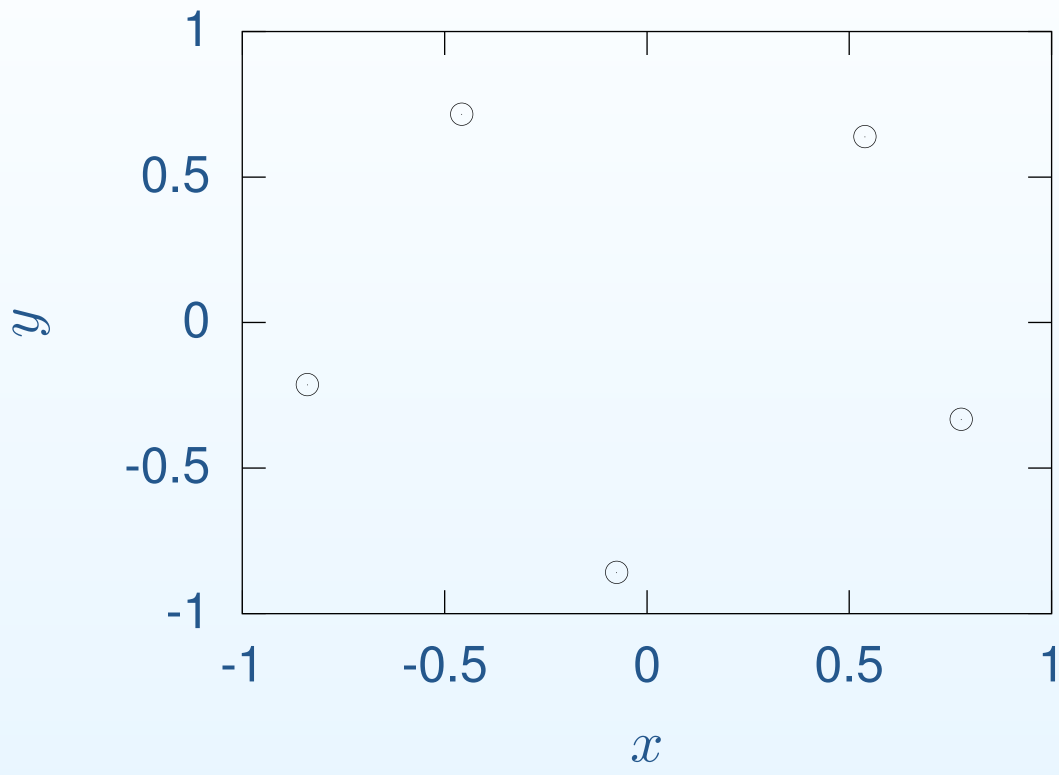

Figure 2. Initial and final configurations.

Formation Control Example Demonstrates Stability and Robustness

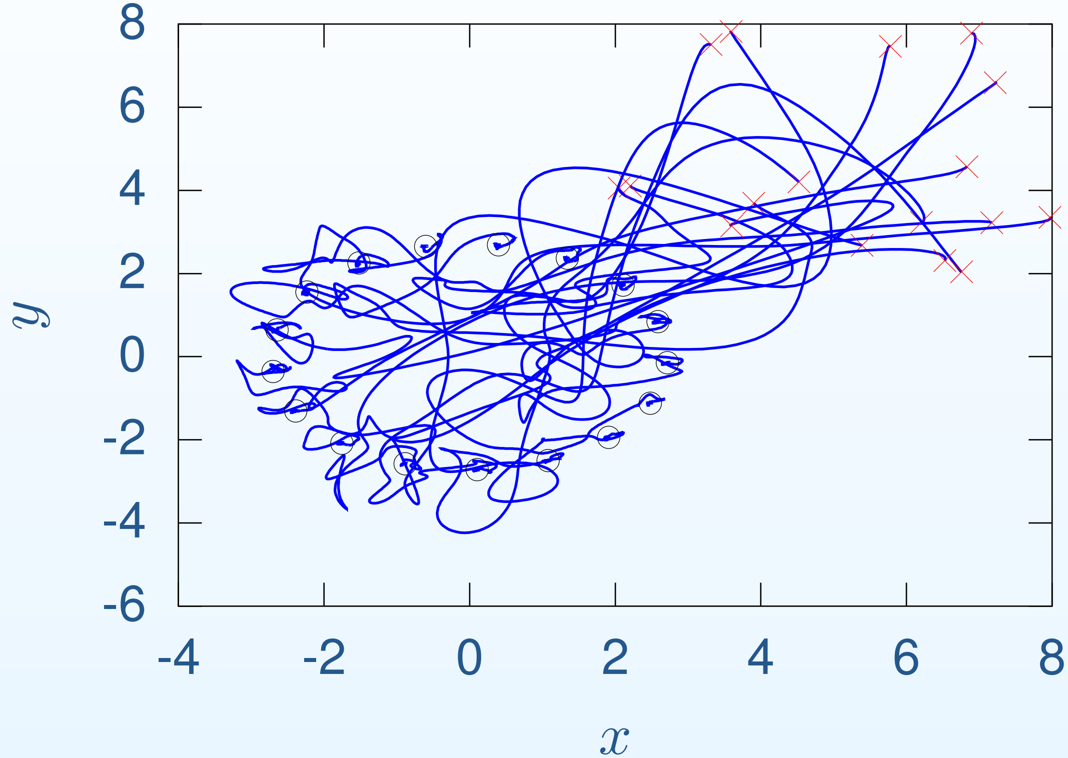

Figure 3. Seventeen-agent formation control.

Formation Control Example Demonstrates Stability and Robustness

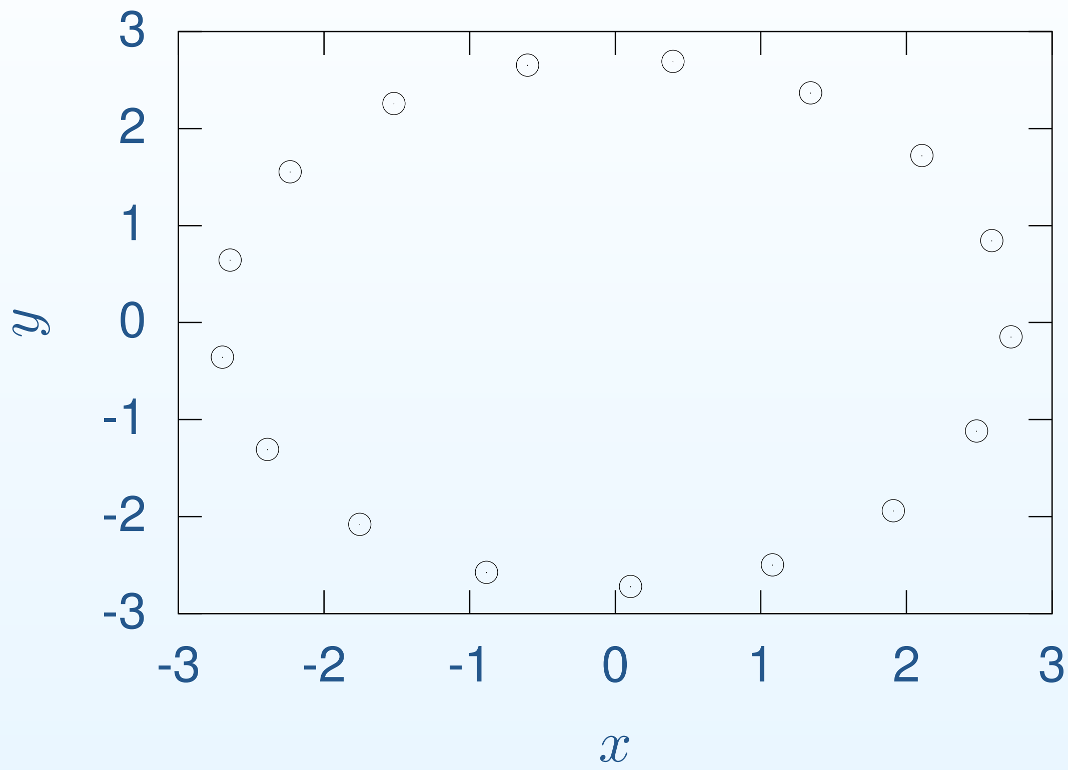

Figure 4. Starting and ending configurations.

Formation Control Example Demonstrates Stability and Robustness

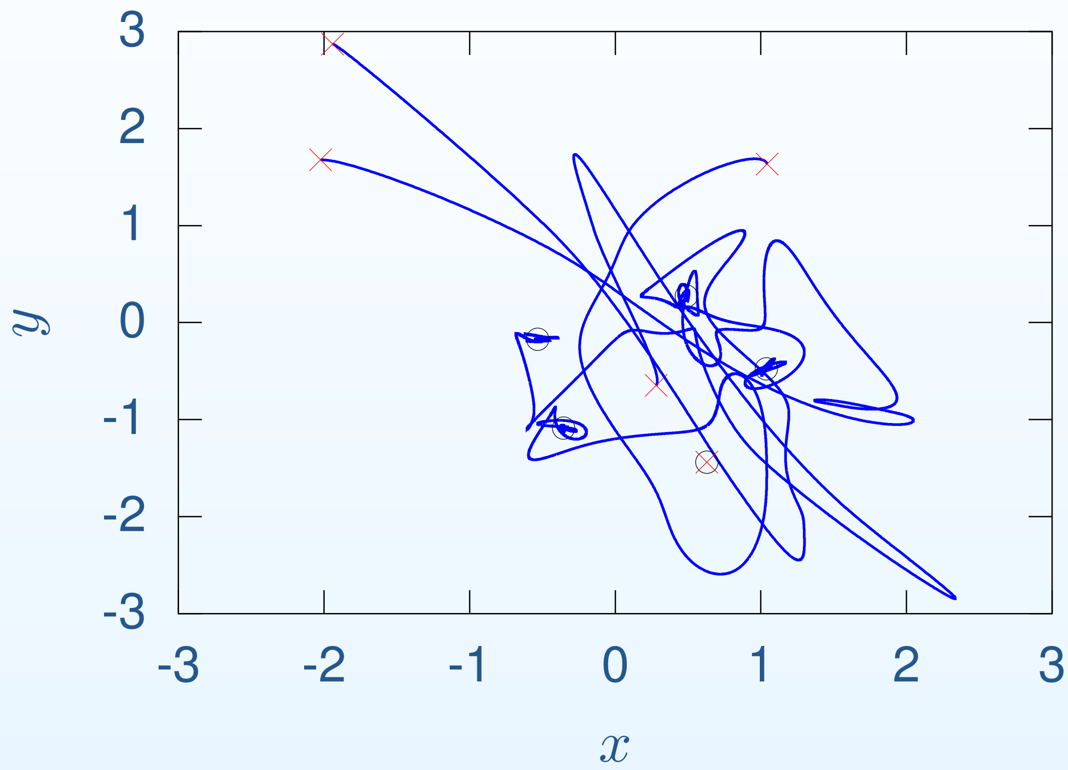

Figure 5. Robust formation stability with agent failure.

Formation Control Example Demonstrates Stability and Robustness

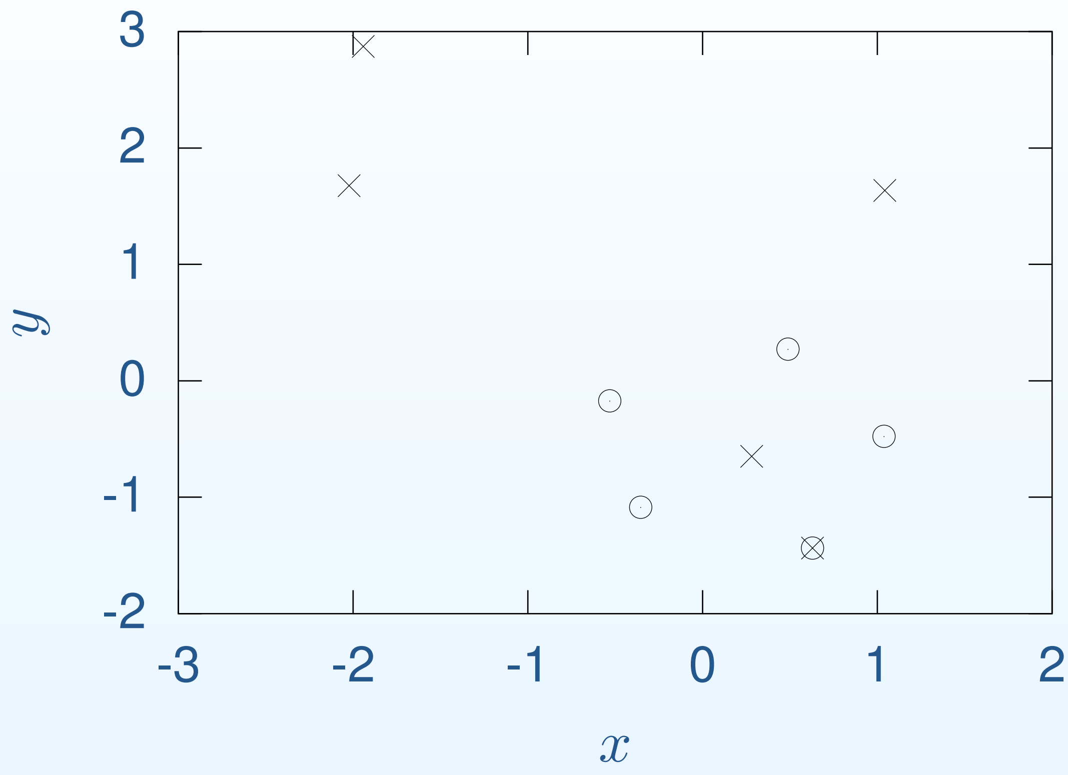

Figure 6. Robust formation stability with agent failure: initial and final configuration.

Note: This is *NOT* Just "Adding Another Stable Component"

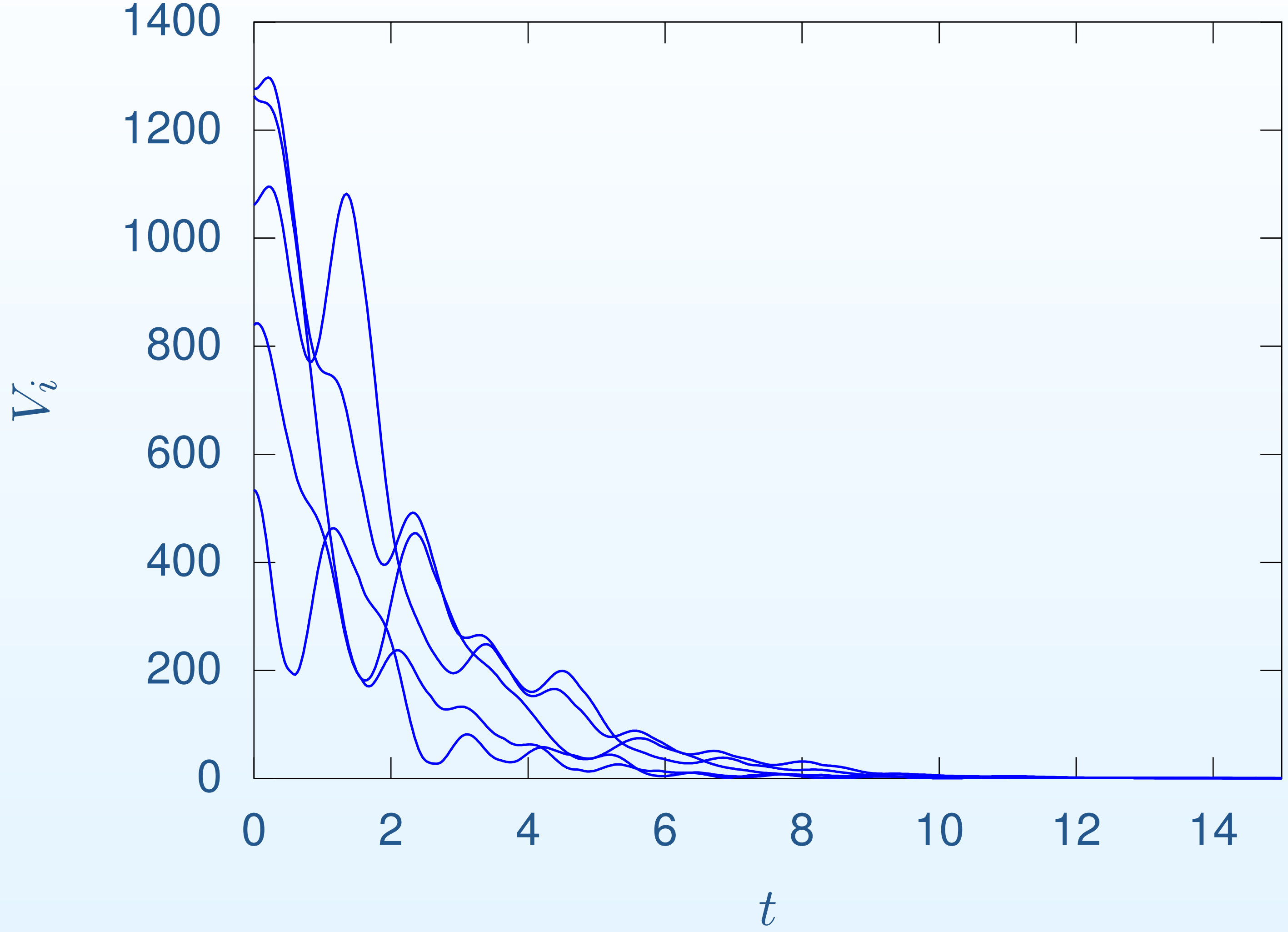

Figure 7. Lyapunov functions for each individual robot a five-vehicle system. Individual robots are *not* stable!

Stability Results Have been Extended to *Approximately* Symmetric Systems

- Of course, in reality, there are no exactly symmetric systems
- Efforts in the past year have been to extend these results to *approximately symmetric systems*
- The “usual” Lyapunov stability: to a point
- The “usual” LaSalle’s invariance principle: stability to a set
- A related issues is *boundedness* either from symmetry-breaking or persistent nonautonomous inputs (surveillance)

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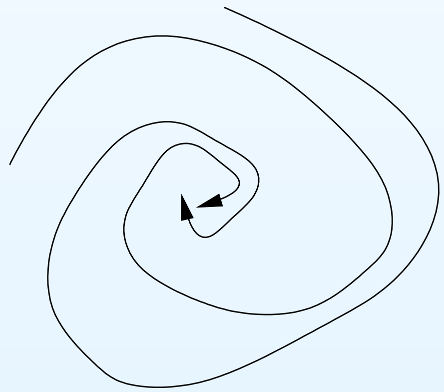

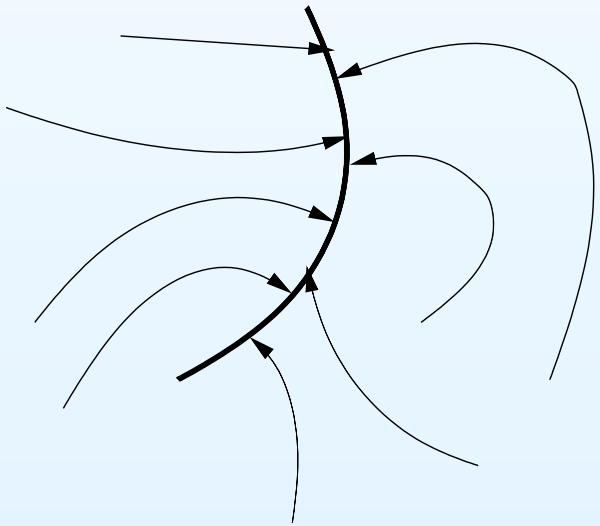

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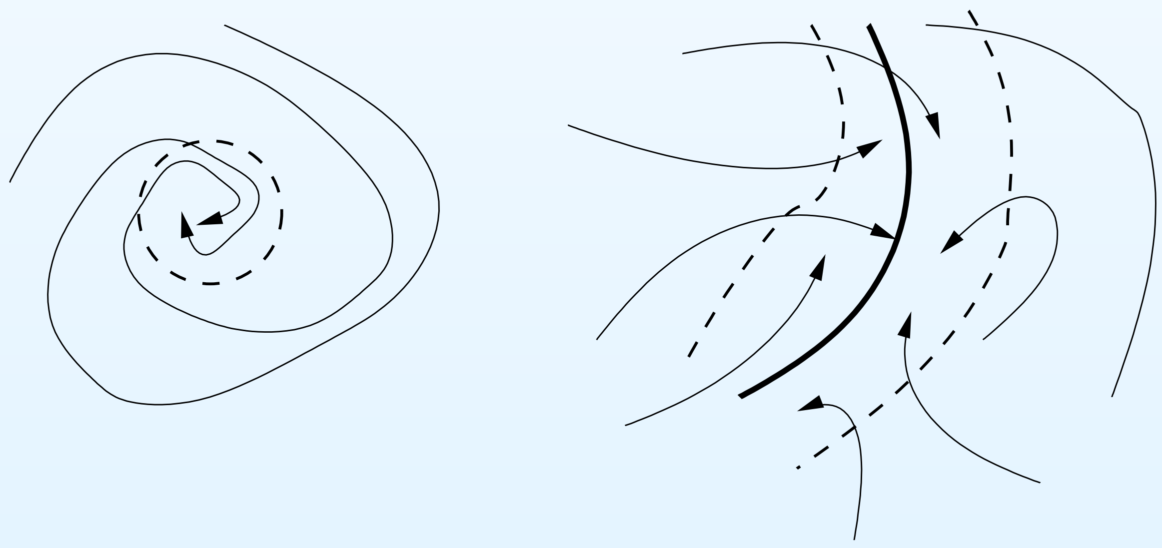

Conditions for Stability to a Set for Autonomous Approximately Symmetric Systems [13]

- Consider approximately symmetric system of the form

$$\begin{aligned}\dot{x}_g(t) &= f_g\left(x_g(t)\right) + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), x_{Xg}(t)\right) \\ &\quad + \hat{f}_g\left(x_g(t)\right) + \sum_{j=1}^{m} \hat{g}_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), x_{Xg}(t)\right) \\ w_g^s(t) &= w_g^s\left(x_g(t)\right)\end{aligned}$$

- Idea: $\dot{V}$ for the symmetric part is more negative than any possible contribution from $\hat{f}(x) + \hat{g}(x)u$.

Conditions for Stability to a Set for Autonomous Approximately Symmetric Systems [13]

PROPOSITION: If

- $d\left(x, dV_0\right)$ represents the distance from a point x to the set of points where $\dot{V}$ for the symmetric system equals zero and
-

$$\frac{\partial V_G}{\partial x_g}(x_G)\left(f_g(x_g)+\sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right) \leq c_1 d^2\left(x, dV_0\right)$$

$$\left\|\frac{\partial V}{\partial x_G}\right\| \leq c_2 d\left(x, dV_0\right),$$

$$\left\|\hat{f}\left(x_g\right)+\hat{g}\left(x_g\right)u\left(x_g, x_{Xg}\right)\right\| \leq c_3 d\left(x, dV_0\right)$$

- Then if $c_2 c_3 / c_1 < 1$ any solution starting in Ω_G approaches the largest invariant set in the set of points where $\dot{V}_g = 0$ as $t \rightarrow \infty$.
- Applies to entire equivalence class.

Formation Control Demonstration of Stability of Approximately Symmetric System

$$\frac{d}{dt}\begin{bmatrix} x_i \\ \dot{x}_i \\ y_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_i \\ 0 \\ \dot{y}_i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k_i\dot{x}_i \\ 0 \\ k_i\dot{y}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{i,1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{i,2}$$

Formation Control Demonstration of Stability of Approximately Symmetric System

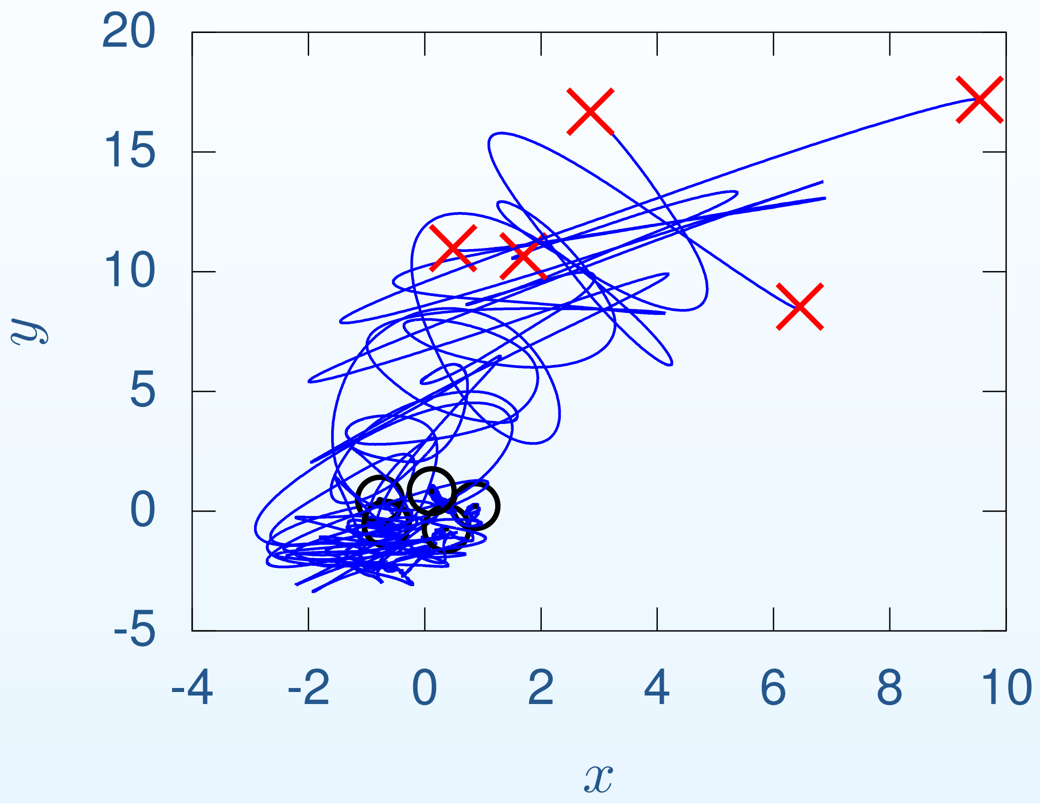

Figure 8. Trajectories for distributed control for an approximately symmetric five-vehicle system.

Formation Control Demonstration of Stability of Approximately Symmetric System

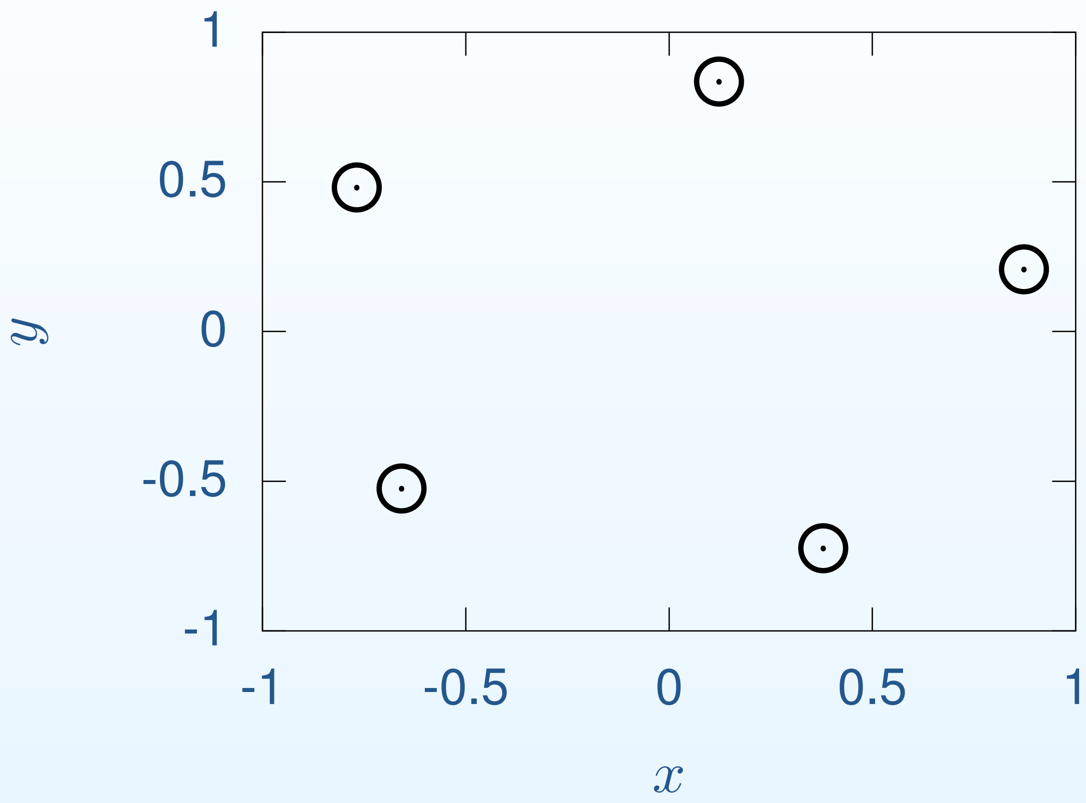

Figure 8. Final formation for distributed control for an approximately symmetric five-vehicle system.

Formation Control Demonstration of Stability of Approximately Symmetric System

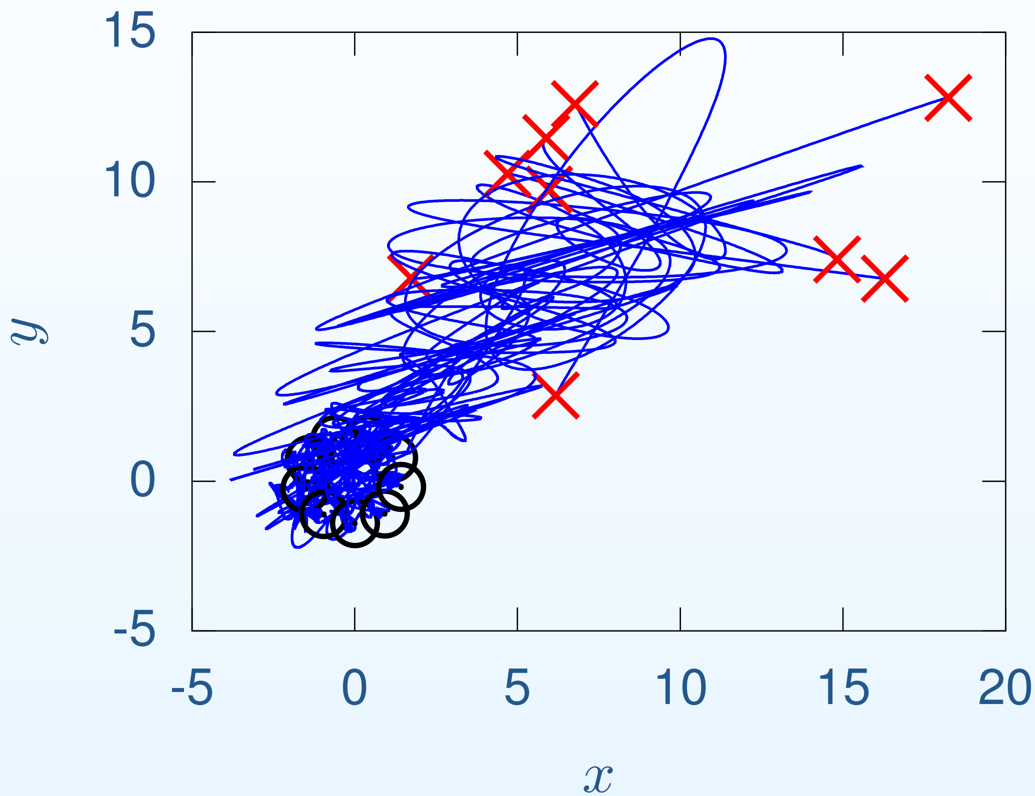

Figure 8. Trajectories for distributed control for an approximately symmetric nine-vehicle system.

Formation Control Demonstration of Stability of Approximately Symmetric System

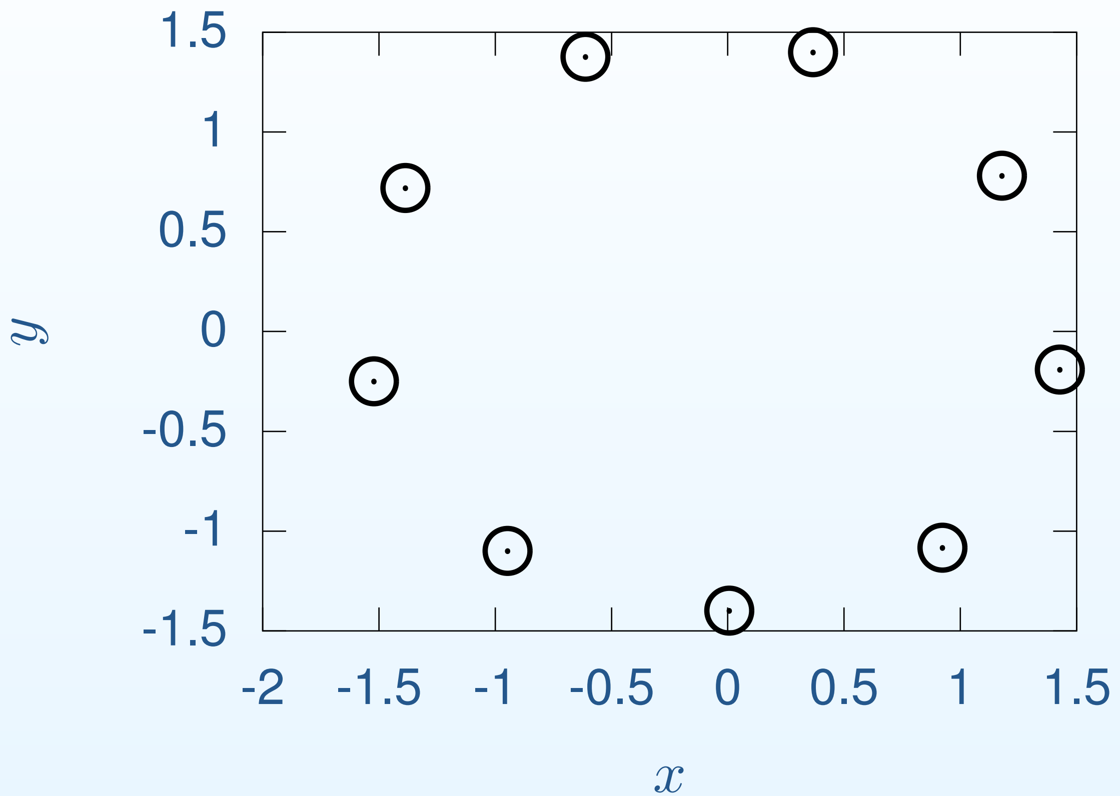

Figure 8. Final formation for distributed control for an approximately symmetric nine-vehicle system.

Bounds on Solutions for Autonomous Approximately Symmetric Systems [3]

- The previous result was when stability of the symmetric part was “stronger” than the symmetry-breaking, so all solutions still converged to the equilibrium.
- Now we consider cases when that is not true, but we know a bound on the symmetry-breaking.
- This leads to a bound on the steady-state solutions from the equilibrium.
- Example (consensus-type problem):
  - Symmetric system:
$$\dot{x} = k \sum_{j \in \mathcal{N}} (x_j - x_i)^3$$
  - Break symmetry but keep symmetric “kernel”
$$\dot{x} = k \sum_{j \in \mathcal{N}} (x_j - x_i)^3 + k \tan^{-1}(x_2)\,.$$

Solutions for Autonomous Approximately Symmetric Systems can be Bounded

PROPOSITION: Given an approximately symmetric system assume the corresponding symmetric system satisfies the hypotheses of the symmetric system stability theorem and that

$$\|p_G(x_G)\| < \delta.$$

Then for any initial condition satisfying $\|x_G\,(t_0)\| < \delta$, the solutions of the approximately symmetric system satisfy

$$\|x_G(t)\| < \delta \tag{2}$$

for all $t \geq t_0$. Furthermore, solutions to any equivalent approximately symmetric system also satisfies Equation 2.

Solutions for Autonomous Approximately Symmetric Systems can be Bounded

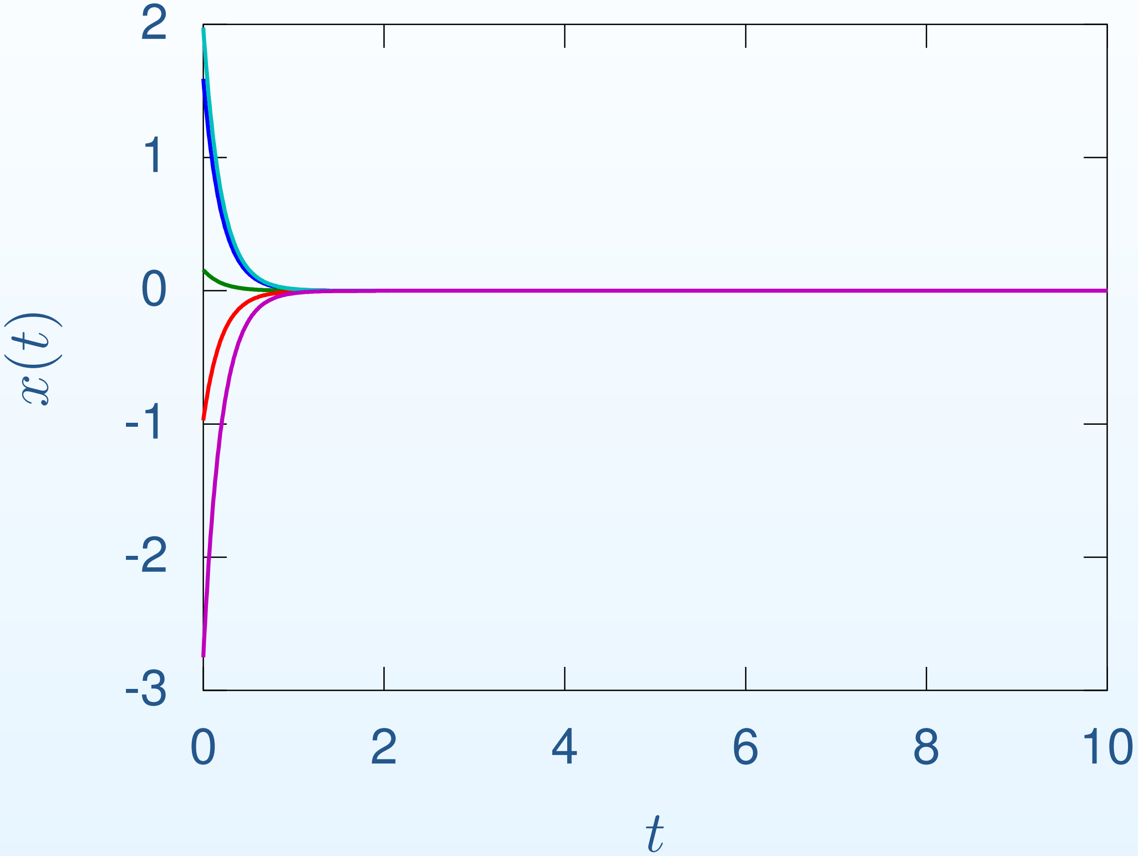

Figure 9. Asymptotic stability for corresponding symmetric five-agent system.

Solutions for Autonomous Approximately Symmetric Systems can be Bounded

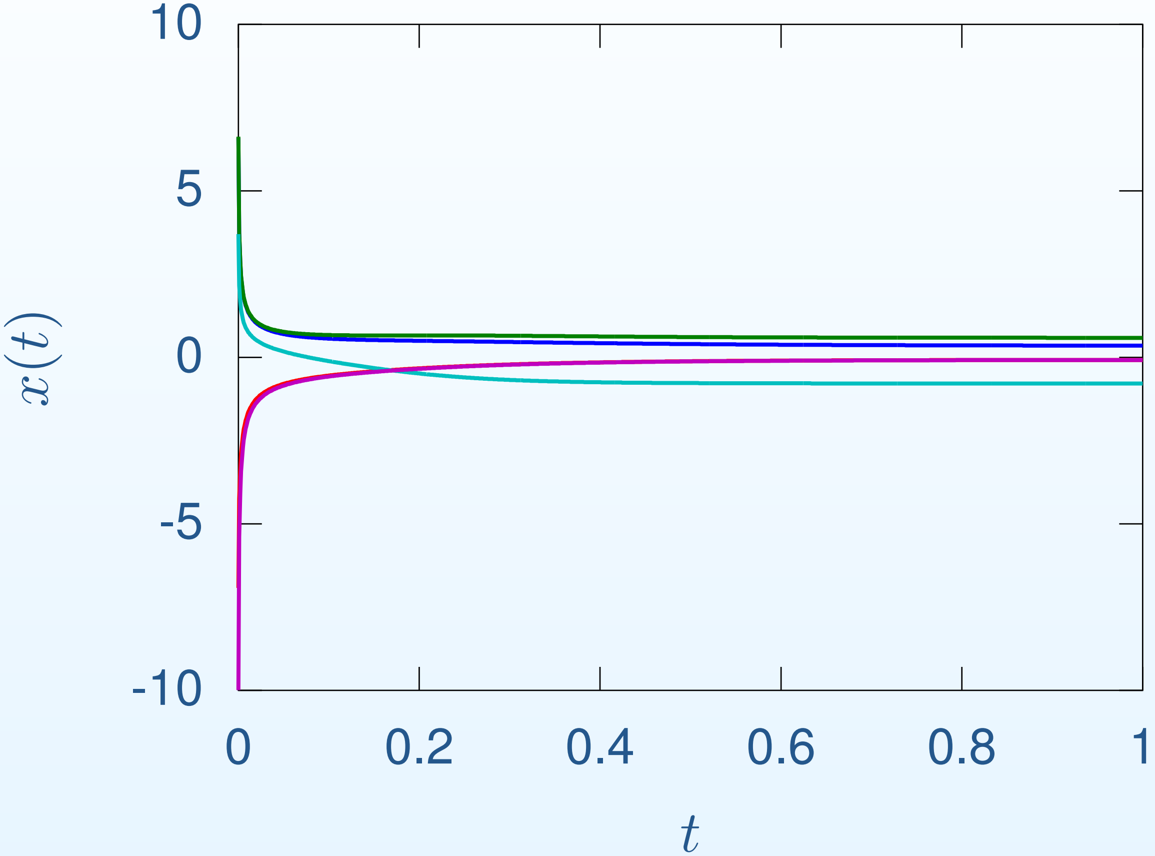

Figure 9. Stability bound for approximately symmetric five-agent system.

Solutions for Autonomous Approximately Symmetric Systems can be Bounded

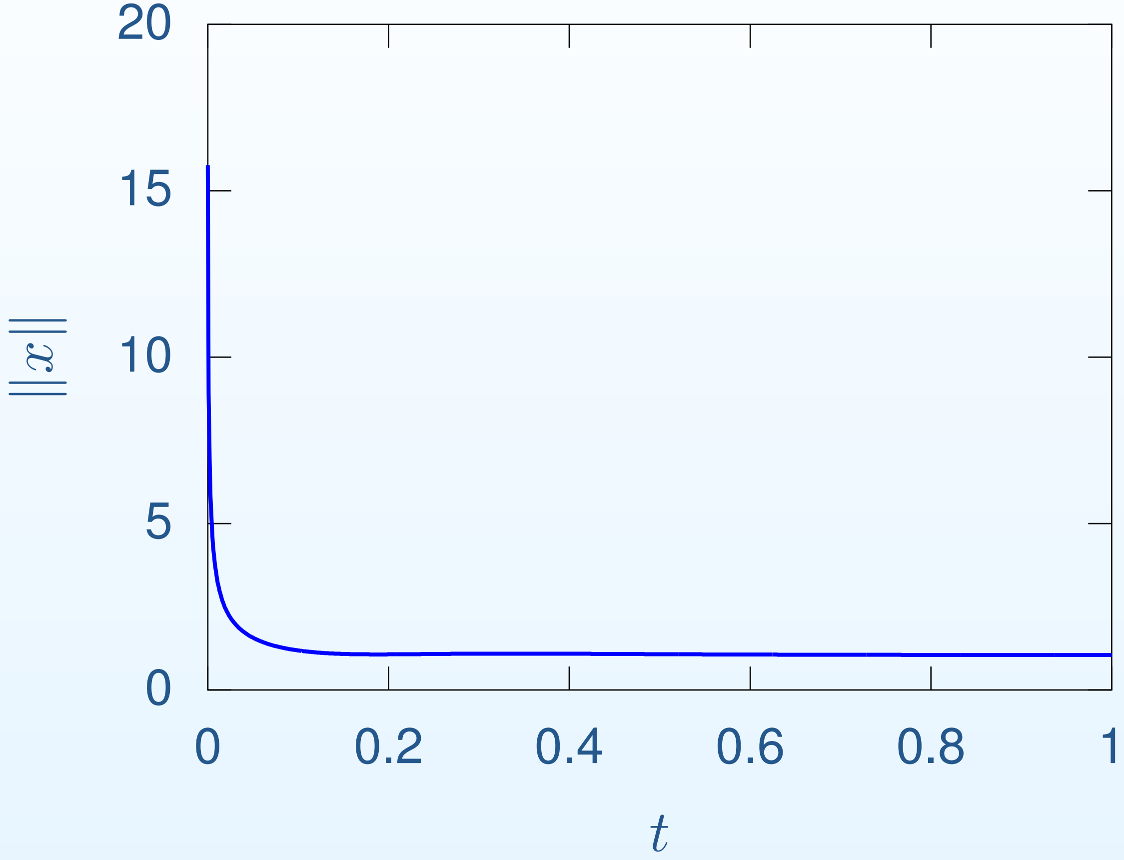

Figure 9. Norm of solution for approximately symmetric five-agent system.

Bound on Solutions Works for Entire Equivalence Class

Fifteen agent system (perturb 1, 2 and 4):

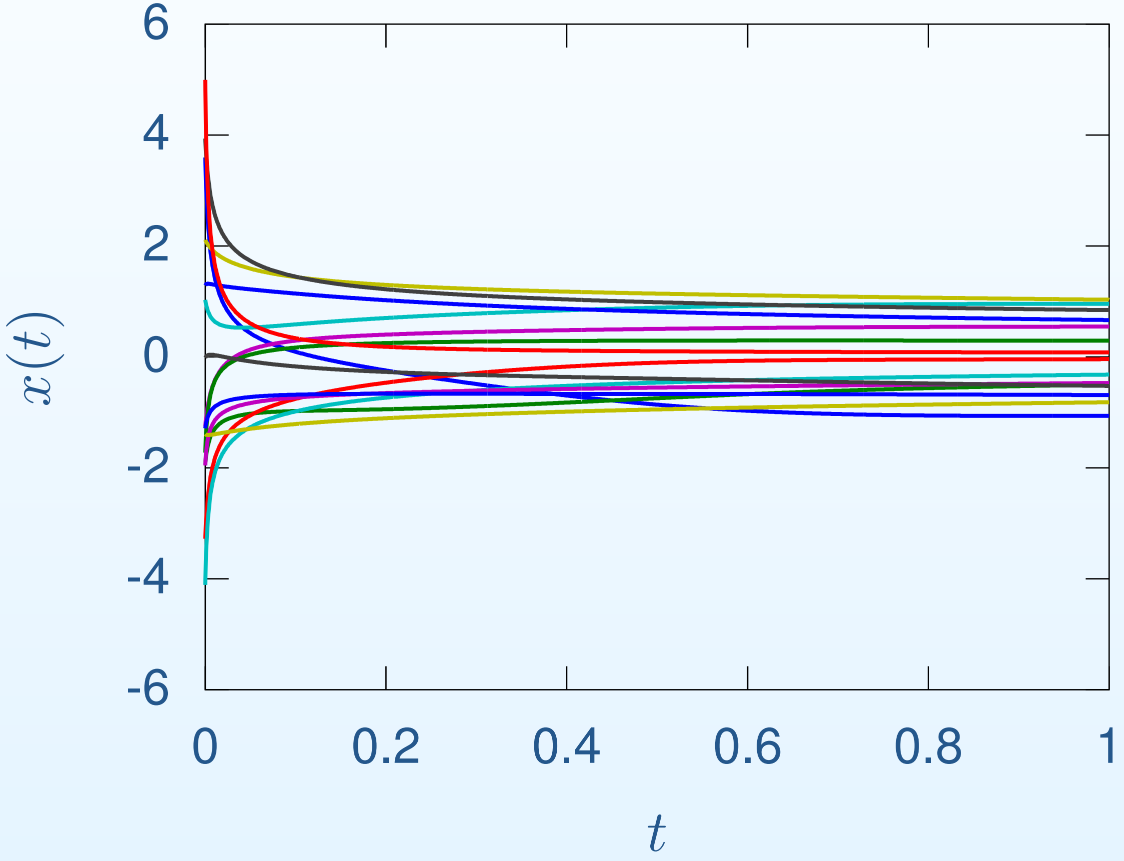

Figure 10. Solution for approximately symmetric fifteen-agent system.

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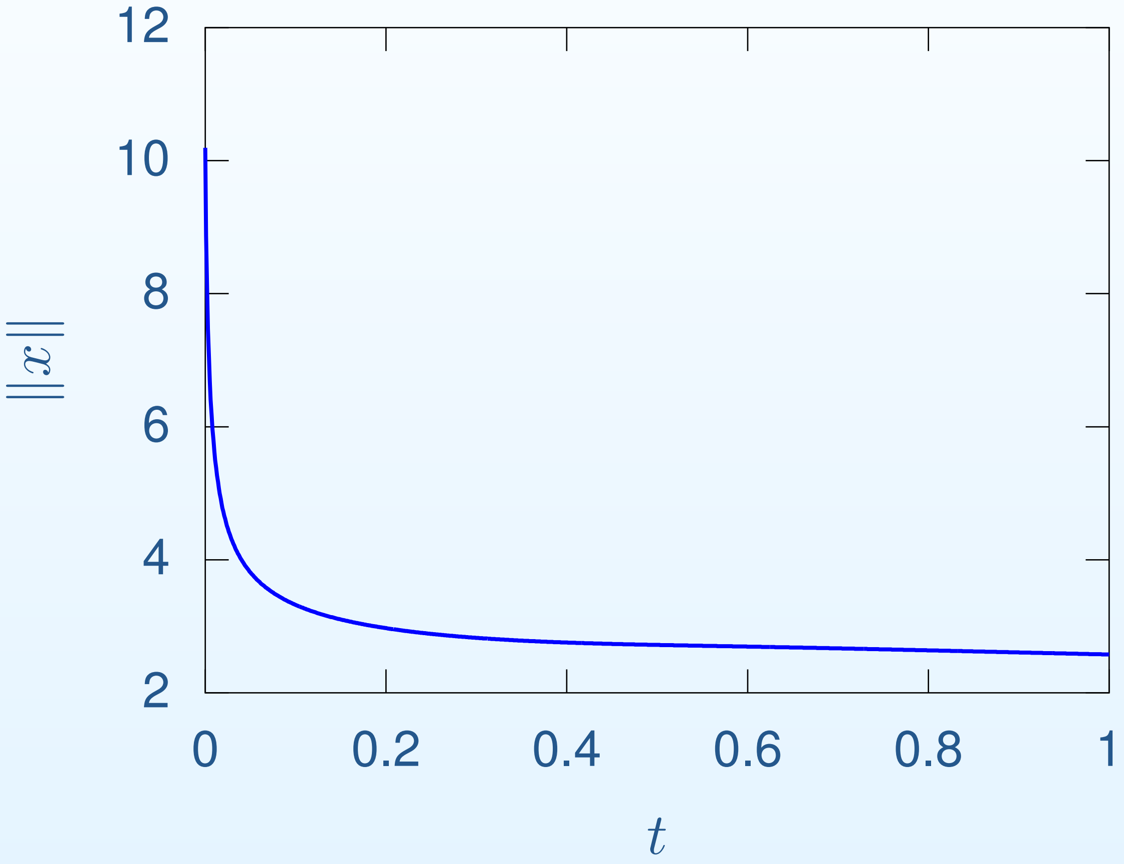

Figure 10. Norm of solution for approximately symmetric fifteen-agent system.

Symmetric Systems with Persistent Nonautonomous Inputs have Bounded Solutions

- In contrast to previous results, this focuses on non-symmetric inputs.
- We consider systems of the form

$$
\begin{aligned}
\dot{x}_g(t) &= f_g\left(x_g(t)\right) \\
&\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), v_g^{s_1}(t), \ldots, v_g^{s_{|X|}}(t)\right) \\
&\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) \hat{u}_{g,j}(t) \\
w_g^s(t) &= w_g^s\left(x_g(t)\right),
\end{aligned}
$$

- Consider the same example as before, but with input term

$$
\dot{x} = k \sum_{j \in \mathcal{N}} \left(x_j - x_i\right)^3 + k_i \sin \omega_i t
$$

Symmetric Systems with Persistent Nonautonomous Inputs have Bounded Solutions

PROPOSITION: Given

- a symmetric system
- $\left\|\sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right)\hat{u}_{g,j}(t)\right\| < c$ for all $g \in \hat{G}$, and,
- for **any one of the** $g \in G$,

$$\frac{\partial V_G}{\partial x_g}(x)\left[f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right] \leq -W_4(x) - c\left\|\frac{\partial V_g}{\partial x_g}(x)\right\|$$

  for all $x_G \in \{\mathcal{D}_G | \, \|x_G\| > \mu_G\|$ where $W_4(x)$ is a positive definite function.
- Then the solutions are bounded and ultimately bounded with computable bounds.

The Nonlinear Consensus Problem Illustrates these Results

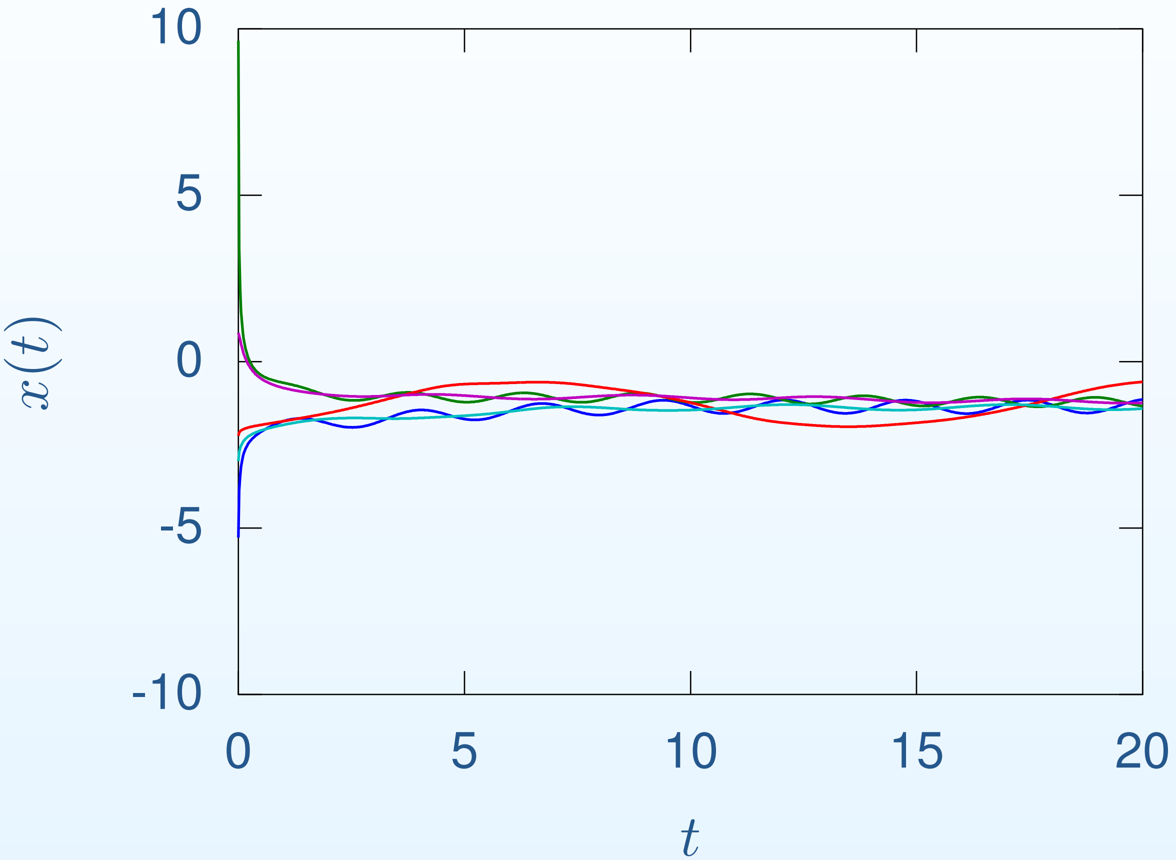

Figure 11. Bounded solutions for five agent system.

The Nonlinear Consensus Problem Illustrates these Results

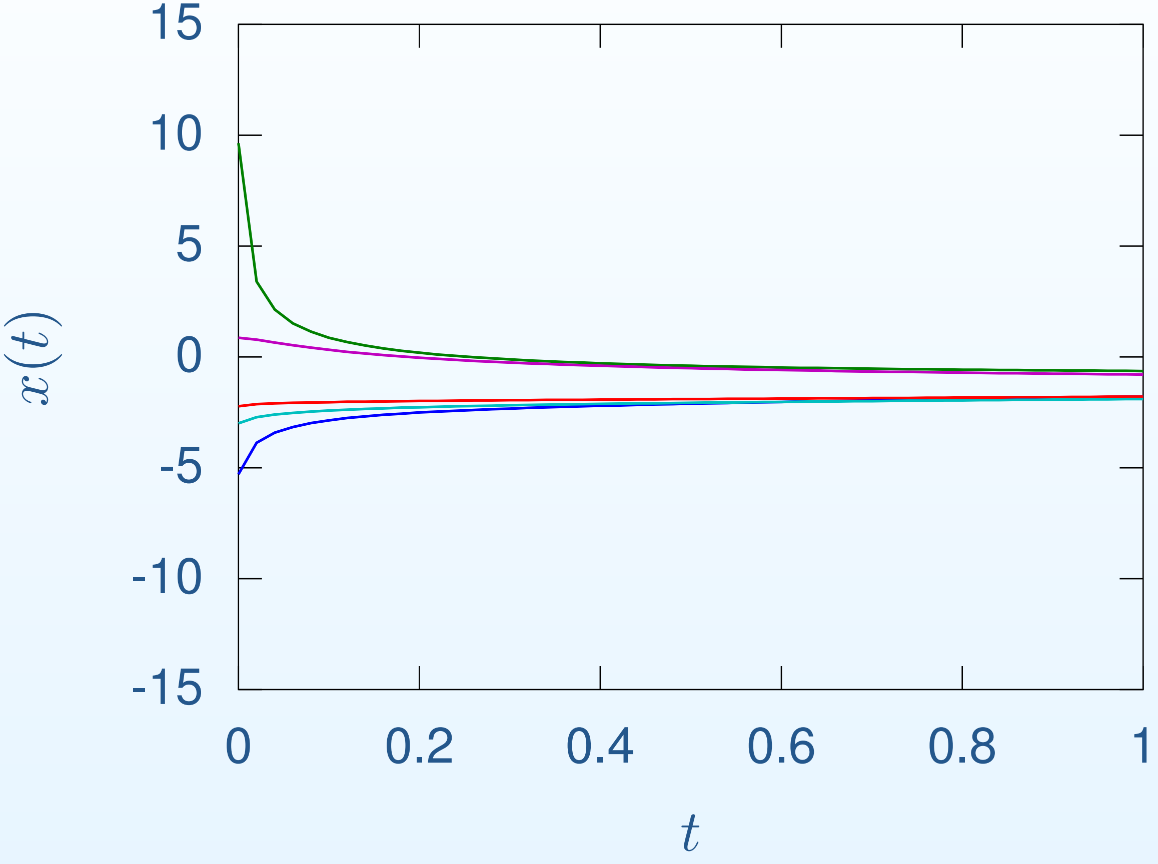

Figure 11. Early solution for bounded five agent system.

The Nonlinear Consensus Problem Illustrates these Results

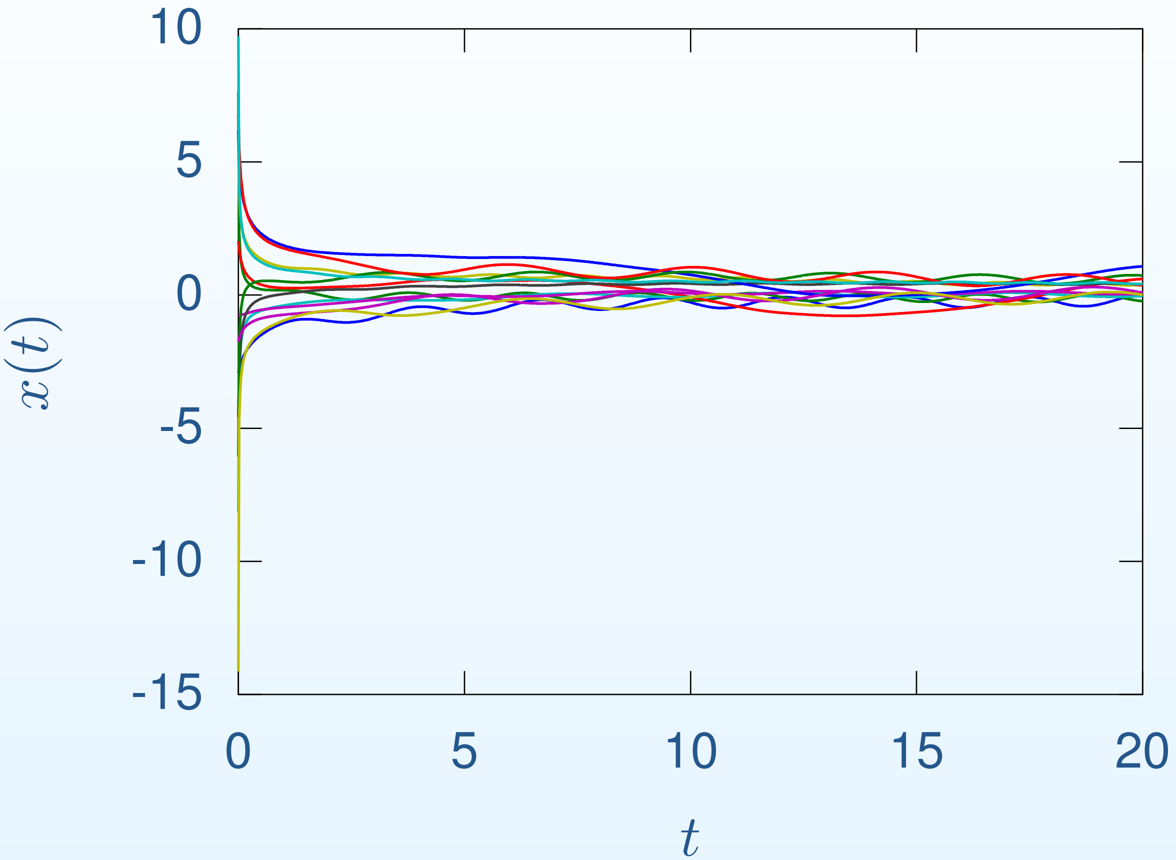

Figure 11. Bounded solutions for thirteen agent system.

Conclusions and Future Work

- This year's work focused on extending the symmetry results from last year to *approximately symmetric systems*.
- This is very useful for realistic systems.
- Extensions in four directions:
  - Approximately symmetric autonomous systems (modeling errors):
    - maintaining stability of equilibrium
    - computing bounds on solutions.
  - Systems with persistent non-symmetric inputs:
    - boundedness of solutions about equilibrium
    - (in progress) boundedness of solutions about sets.
- Current efforts
  - connections to passivity and dissipativity
  - connections to applications (ACC, power grid)

References

References

[1] Raffaello DÁndrea and Geir E. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, 48(9):1478–1495, September 2003.

[2] Bill Goodwine. Compositional boundedness of solutions for symmetric nonautonomous control systems. December 2013. In Submitted to 52nd IEEE Conference on Decision and Control.

[3] Bill Goodwine. Nonlinear control and stability of approximately symmetric systems of coordinating robotic agents. November 2013. Submitted to 2013 IEEE/RSJ International Conference on Intelligent Robots and Systems.

[4] Bill Goodwine and Panos Antsaklis. Multiagent compositional stability exploiting system symmetries. *Automatica*, 2013. Accepted for publication.

[5] M. Brett McMickell and Bill Goodwine. Reductionand controllability of symmetric distributed systems with robotic applications. In *International Conference on Intelligent Robotics and Systems*, volume 3, pages 1232–1236. IEEE/RSJ, October 2001.

[6] M. Brett McMickell and Bill Goodwine. Reduction and controllability of robotic systems with drift. In *Proceedings of the 2002 IEEE International Conference on Robotics and Automation*, May 2002.

[7] M. Brett McMickell and Bill Goodwine. Motion planning for symmetric distributed robotic systems. In *2003 IEEE International Conference on Robotics and Automation*, September 2003.

[8] M. Brett McMickell and Bill Goodwine. Reduction and nonlinear controllability of symmetric distributed systems. *International Journal of Control*, 76(18):1809–1822, 2003.

[9] M. Brett McMickell and Bill Goodwine. Motion planning for nonlinear symmetric distributeed robotic systems. *International Journal of Robotics Research*, 26(10):1025–1041, October 2007.

[10] M. Brett McMickell, Bill Goodwine, and Luis Antonio Montestruque. Micabot: A robotic platform for large-scale distributed robotics. In *2003 IEEE International Conference on Robotics and Automation*, September 2003.

[11] Reza Olfati-Saber and Richard M. Murray. Distributed cooperative control of multiple vehicle formations using structural potential functions. In *Proceedings of the 2002 IFAC World Congress*, July 2002.

[12] Benjamin Recht and Raffaello DÁndrea. Distributed control of systems over discrete groups. *IEEE Transactions on Automatic Control*, 49(9):1446–1452, September 2004.

[13] Compositional stability of approximately symmetric systems: Initial results. Bill goodwine. In *In Proceedings of the 21st Mediterranean Conference on Control and Automation*, June 2013. Accepted for publication and presentation.