Sufficient Statistics for Multi-Agent Systems

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Stochastic decision problems

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u = \mu(y) \end{array}$$

- given joint pdf of x, y, find best μ
- *e.g.*, estimation: with $c = \|x - u\|$
- can generate y, x with a model, *e.g.*, $y = Ax + w$
- hypothesis testing, classification, detection, decision, etc.,

Sufficient statistics

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u = \mu(y) \end{array}$$

$s = g(y)$ is called *sufficient* for $x \mid y$ if

$$y \perp x \mid s$$

- equivalently $\mathrm{prob}(x \mid s, y)$ does not depend on y
- conditional distribution $x \mid y$ depends only on s
- Fisher 1922, Kolmogorov 1942
- *optimal policy* has the form $u = \mu(s)$

Examples: Sufficient statistics

- Gaussian noisy measurements $y_i = x + w_i$ then $s = \sum_i y_i$
- multiplicative uniform noise: $y_i = x w_i$ then $s = \max_i y_i$
- y_i is Bernoulli with $\mathrm{prob}(y_i = 1 \mid x) = x$, then $s = \sum_i y_i$
- if $y = Ax + w$ and w is Gaussian, then $s = A^T y$
- y_i has discrete uniform distribution on $[0, x]$, then $s = \max_i y_i$ called *German tank problem*
- many others ...

Team decision problems

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x, u_1, \dots, u_n) \\ \text{subject to} & u_i = \mu_i(y_i) \end{array}$$

- given joint distribution $x, y_1, \dots, y_n$, find n *policies* μ_i
- Marschak, 55, Radner, 62, Tsitsiklis and Athans, 85
- decentralized control

Optimization: Single-player

$$\begin{array}{ll} \text{minimize} & \sum_{xyu} c_{xu} p_{xyu} \\ \text{subject to} & p_{xyu} = q_{xy} K_{yu} \\ & K_{yu} \text{ binary, stochastic} \end{array}$$

- LP relaxation: if p_{xyu} is in the convex hull, then

$$\boxed{u \perp x \mid y}$$

- equivalently: u generated by randomized policy $u = \mu(y, w)$

Optimization: Multi-player

$$\begin{array}{ll} \text{minimize} & \sum_{xyu} c_{xu} p_{xyu} \\ \text{subject to} & p_{xyu} = q_{xy} K^1_{y_1 u_1} \dots K^n_{y_n u_n} \\ & K^i_{y_i u_i} \text{ binary, stochastic} \end{array}$$

- if p_{xyu} is in the convex hull, then call $u_1, \dots, u_n$ a *team decision*, and write

$$\boxed{u_1, \dots, u_n \perp\!\!\!\perp x \mid y_1, \dots, y_n}$$

- equivalently: u generated by common randomness $u_i = \mu_i(y_i, w)$
- $u \perp\!\!\!\perp x \mid y$ implies $u \perp x \mid y$

Relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x,u) \\ \text{subject to} & u_1, \dots, u_n \perp\!\!\!\perp x \mid y_1, \dots, y_n \end{array}$$

Multi-player sufficient statistics

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c(x, u_1, \dots, u_n) \\ \text{subject to} & u_i = \mu_i(y_i) \end{array}$$

if $s_i = g_i(y_i)$, then $s_1, \dots, s_n$ are called *team sufficient* for $x \mid y_1, \dots, y_n$ if

$$y_1, \dots, y_n \perp\!\!\!\perp x \mid s_1, \dots, s_n$$

theorem: if s is team sufficient, then there exists an optimal policy of the form

$$\boxed{u_i = \mu_i(s_i)}$$

Multi-player sufficient statistics

- $s_1, \dots, s_n$ is team sufficient if $y_1, \dots, y_n \perp\!\!\!\perp x \mid s_1, \dots, s_n$
- then there is a deterministic optimal controller $u_i = \mu_i(s_i)$
- *optimal* and *deterministic* even though s is defined in terms of
  - convex hull of feasible distributions
  - randomized policies

Example: Two players

suppose

s_1 is sufficient for $x, y_2 \mid y_1$
s_2 is sufficient for $x, y_1 \mid y_2$

then s_1, s_2 is team sufficient for $x \mid y_1, y_2$

for example, if x, y_1, y_2 are jointly Gaussian, then

$$s_1 = \mathbf{E}\left(\begin{bmatrix} x \\ y_2 \end{bmatrix} \middle| \, y_1 \right) \qquad s_2 = \mathbf{E}\left(\begin{bmatrix} x \\ y_1 \end{bmatrix} \middle| \, y_2 \right)$$

Example: Triangular

- measurements: $y_1 = z_1$ and $y_2 = (z_1, z_2)$
- suppose

  r_1 is sufficient for $r_2 \mid z_1$
  r_2 is sufficient for $x \mid z_1, z_2$

- then s_1, s_2 are team sufficient statistics for $x \mid y_1, y_2$, where

$$s_1 = r_1$$
$$s_2 = (r_1, r_2)$$

- Gaussian case

$$s_1 = \mathbf{E}(x \mid z_1)$$
$$s_2 = (\mathbf{E}(x \mid z_1), \mathbf{E}(x \mid z_1, z_2))$$

Algebraic rules

For $s = g(y)$ and $t = h(y)$, most of these generalize to $\perp\!\!\!\perp$

$$\begin{array}{rcl} y \perp x \mid f(s) & \implies & y \perp x \mid s \\ y \perp x \mid s & \implies & y \perp f(x) \mid s \\ y \perp x \mid s & \implies & y \perp f(x,s) \mid s \\ y \perp z \mid s \text{ and } y \perp x \mid s & \implies & y \perp x, z \mid s \\ y \perp x \mid s \text{ and } s \perp x \mid t & \implies & y \perp x \mid t \end{array}$$

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