A Mathematical Framework for Modelling Dynamics and Information Flow in a Network

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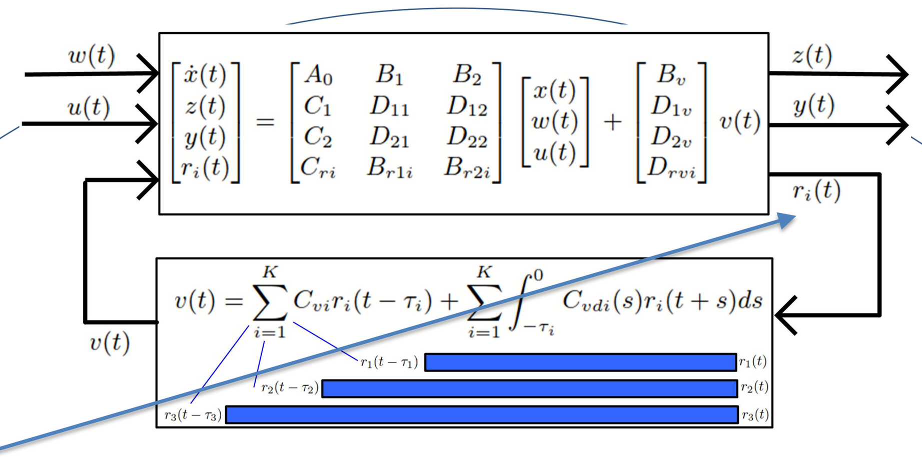
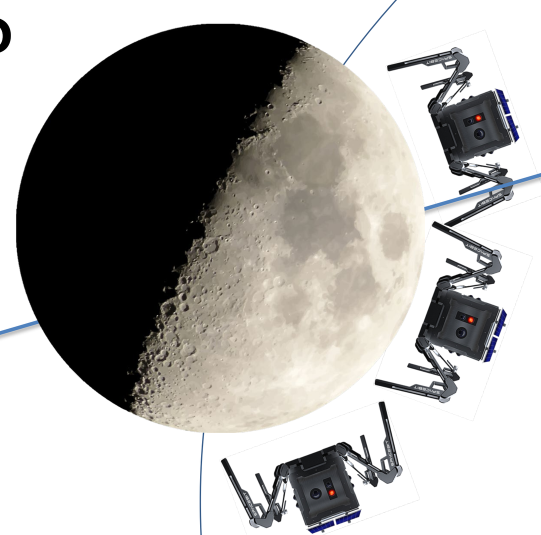
http://control.asu.edu/NSF_1739990.htm

Modelling Information Flow in Networks:

- How to efficiently represent the flow of information in a network of interconnected systems?
- Algebra is the language of computation





Differential Difference Models:

- Isolate heterogeneous flows of information, $r_i(t)$
- A precursor to the PIE

The New Matrix: PI Operators

- PI operators are closed under composition, concatenation, adjoint, addition, and sometimes inversion

Definition of a 4-PI Operator $\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix}\right): \mathbb{R}\times L_2 \to \mathbb{R}\times L_2$

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix}\begin{bmatrix} x \\ \mathbf{\Phi}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^{0} Q_1(s)\mathbf{\Phi}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s)\end{bmatrix}.$$

4-PI Operators include a 3-PI Operator, Defined as:

$$\left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s) := R_0(s)\mathbf{\Phi}(s) + \int_{-1}^{s} R_1(s,\theta)\mathbf{\Phi}(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\mathbf{\Phi}(\theta)d\theta$$

"Change is the only constant in life" - Heraclitis

- The PIE state is change in state:

$$\int_s^0 \dot{\phi}_s(t,\theta)d\theta = \dot{r}(t) - \frac{1}{\tau}\phi_s(t,s)$$

- State no longer needs to be continuous or differentiable

Partial Integral Equation (PIE) Models: Just another state…

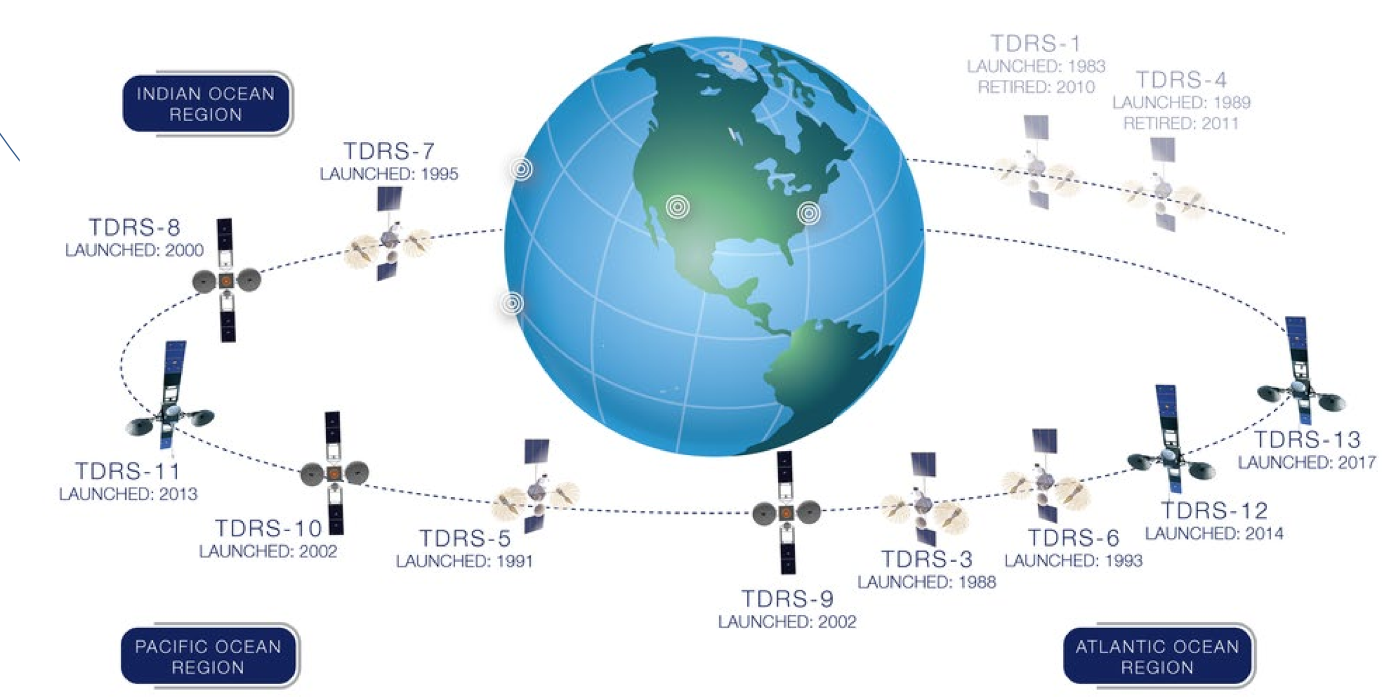
- Integrates distributed and ODE states in a single algebraic representation using only bounded PI operators
- Perfect for simulation, analysis, and controller synthesis

The Class of Partial Integral Equation (PIE) Systems:

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{B}_{T_1}\dot{w}(t) + \mathcal{B}_{T_2}\dot{u}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11}w(t) + \mathcal{D}_{12}u(t)\\ y(t) &= \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t) + \mathcal{D}_{22}u(t)\end{aligned}$$

PIETOOLS

http://control.asu.edu/pietools



PIETOOLS Network User Interface:

- PIETOOLS 2021: A comprehensive toolbox for handling of information flow in networks
- Input a network as Delay-Differential Equation (DDE), Differential Difference Equation (PDF), Partial Integral Equation (PIE) or just use the Graphical User Interface (GUI)

PIETOOLS: Converting Between Representations

- Functions to convert between DDE, DDF, and PIE representations

PIETOOLS: Constructing Minimal Representations

- Functions identify low-dimensional subspaces of the information flow
- Constructs equivalent minimal DDF and PIE representations of flow

The New LMI: Linear PI (LPI) Optimization

- Because PIs are an algebra, we parameterize positive PIs with positive matrices: $Z^*PZ \geq 0$ if matrix $P \geq 0$
- Extends the LMI framework to optimization of PIs

Important LPIs: H∞-Optimal Control and Estimation

- Most LMIs for control of ODEs can be extended to PIEs
- Troublesome bit is T-operator (same as in singular systems)

H_∞-optimal Estimator: $\min_{\gamma,\mathcal{P}} \gamma$

$$\mathcal{P} \succ 0$$

$$\begin{bmatrix} -\gamma I & -\mathcal{D}_{11}^* & -(\mathcal{P}\mathcal{B}_1 + \mathcal{D}_{21}\mathcal{Z})^*\mathcal{T} \\ -\mathcal{D}_{11} & -\gamma I & \mathcal{C}_1 \\ ()^* & \mathcal{C}_1^* & ()^* + \mathcal{T}^*(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\end{bmatrix} \preccurlyeq 0$$

H_∞-optimal controller: $\min_{\gamma,\mathcal{P}} \gamma$

$$\mathcal{P} \succ 0$$

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}_1\mathcal{P} + \mathcal{Z}D_{12}\mathcal{Z})\mathcal{T}^* \\ \mathcal{D}_{11}^* & -\gamma I & \mathcal{B}_1^* \\ ()^* & \mathcal{B}_1 & ()^* + (\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})\mathcal{T}^*\end{bmatrix} \preccurlyeq 0$$

Hey, check this out

PIETOOLS took a network with 400 information flows and reduced them to 10, then found an H_∞-optimal controller in 1.5 minutes! Not bad, right?

Reduced Computation Time:

Ex.	Dimension Size nom	Dimension Size min	CPU seconds nom	CPU seconds min
Ex. 1 (n=5)	60	9	N/A	220.6
Ex. 1 (n=10)	220	19	N/A	9,350
Ex. 2 (n=5)	100	5	N/A	2.42
Ex. 2 (n=10)	400	10	N/A	94.7
Ex. 3	8	2	22.56	.332
Ex. 4	10	5	147.3	4.915

Table: Computation times for nominal and minimal realizations. Times are H_∞-control for Exs. 1 and 2 and stability analysis for Exs. 3 and 4.

Spiderbots on the Moon:

- Cislunar Autonomous Positioning System (CAPS) provides GPS-like positioning and packet routing using ad-hoc network connections between assets in lunar orbit.
- NASA's Tracking and Data and Relay System (TDRS) constellation provides data routing relay stations from anywhere in deep space to NASA ground stations.
- The first spiderbot (a 1kg modified cubesat) is scheduled for lunar deployment in Q3 2021 by Spacebit using a ULA Vulcan rocket and Astrobotic Peregrine lander.

Broader Impacts:

- We standardize the representation of information flow in a network
- We created easy-to-use PIETOOLS to maximize both industry and academic impact
- Outreach via workshops, seminars, youtube and website
- Mentoring of high-school research project from underserved communities in Phoenix
- Channels for: Sensors, Commands, Disturbances, States
- Integrates Stability, Optimal Estimation, Optimal Control, etc.
- *Order of Magnitude improvement: Control Networks with 50+ latent communication channels*

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