Congested Equilibria in Traffic Networks: Existence, Stability and Robustness

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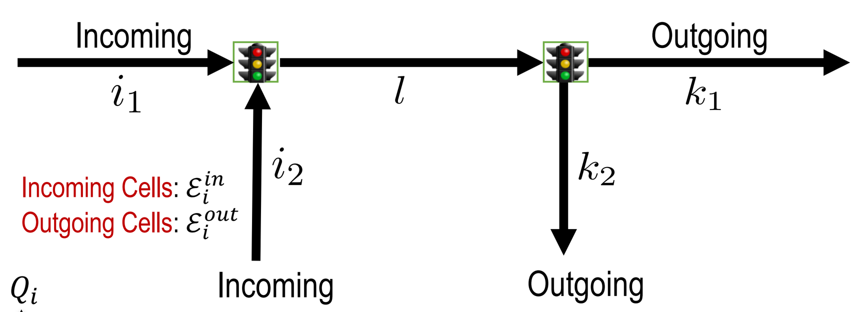
Abstract

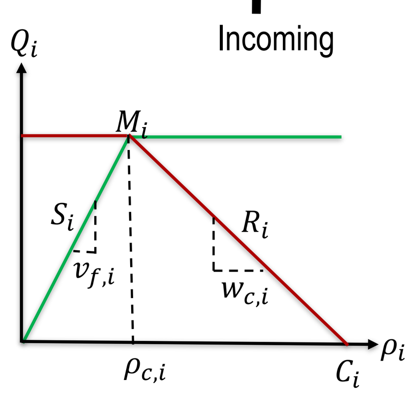
Fluid-like models like the Cell Transmission Model have proven successful in modeling traffic networks. However, given the complexity of the dynamics, it is not surprising that the dynamical properties of these models, especially in congested regimes, are not yet well characterized. My research addresses this gap by proposing a new modeling paradigm where an **analogy between discretized fluid-like traffic flow models and a class of chemical reaction networks** is constructed by suitable relaxations of key conservation laws. This framework allows us to draw upon powerful structural results from chemical reaction network theory to **study** the existence and stability of **congested steady states** in large-scale transportation networks. We show this framework can be used in traffic planning to **identify and eliminate congestion bottlenecks** in large-scale traffic networks.

We propose a **new modeling paradigm for traffic**, that draws upon an **analogy with chemical reaction networks** to understand the creation and propagation of traffic jams.

Cell Transmission Model of Traffic Flow

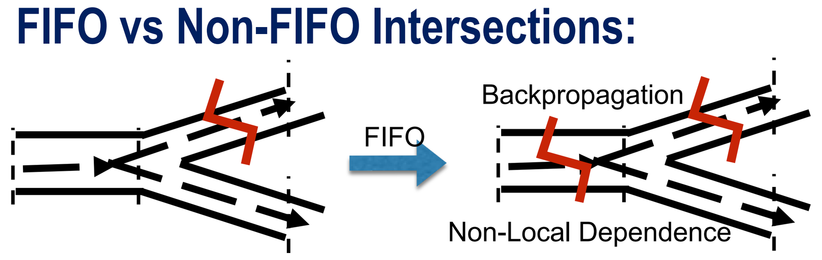


- Network is partitioned into "cells" - roads or sections of roads
- Traffic is a fluid flow between cells



- Every cell i is assigned a "state" - number of vehicles $\rho(t)$, an incoming flow $Q_{i,in}(t)$ and an outgoing flow $Q_{i,out}(t)$.
- Density and flow are related by the fundamental diagram.

FIFO vs Non-FIFO Intersections:



Does congestion on one fork of diverge cause traffic going to the other fork to back up?

System Dynamics:

$$Q_{i,in}(t) = \sum_{k \in \mathcal{E}_i^{in}} \alpha_{k,i}^{u} \min\{\lambda_i(t) + B_{ki} S_k(t) + w_i(t), R_i(t)\}$$

(External baseline inflow: $\lambda_i(t)$; Disturbance: $w_i(t)$)

$$Q_{i,out}(t) = \begin{cases} \sum_{j \in \mathcal{E}_i^{out}} B_{ij} \min\left\{S_i(t), \frac{R_j(t)}{B_{ij}}, \frac{R_k(t)|_{k \in \mathcal{E}_i^{out}, k \neq j}}{B_{ik}}\right\} & \text{FIFO} \\ \sum_{j \in \mathcal{E}_i^{out}} \min\{B_{ij} S_i(t), \alpha_{ij} R_j(t)\} & \text{Non-FIFO} \end{cases}$$

(Turning ratio: B_{ij})

$\rho_i(t+1) = \rho_i(t) + Q_{i,in}(t) - Q_{i,out}(t)$ ⟶ Mass conservation law

$\rho(t+1) = F(\rho(t), u(t), w(t))$ ⟶ Hybrid piecewise affine dynamical system

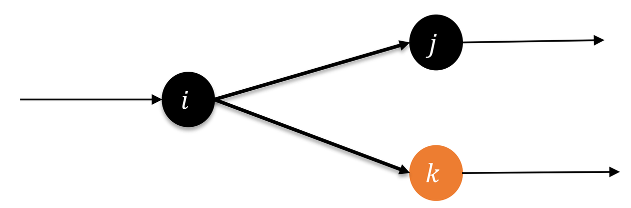
Sivaranjani S was partially supported by the Schlumberger Foundation Faculty for the Future Fellowship.

CPS: Synergy: Collaborative Research: Beyond Stability: Performance, Efficiency, and Disturbance Management for Smart Infrastructure Systems

Analysis of Congested Transportation Networks: Chemical Reaction Network Analogy

- Does the CTM model have congested steady states in which the network can get stuck?
- Can we identify these bottlenecks and how to remove them?

Challenges:



$$Q_{i,out}(t) = Q_{i,j}(t) + Q_{i,k}(t) = \frac{B_{ij}}{B_{ik}} w_{c,k}(C_k - \rho_k) + w_{c,k}(C_k - \rho_k)$$

- Dynamics are non-compartmental - outflow from a cell depends on the density of downstream cells (back-propagation of congestion).
- Dynamics have non-local dependence of flows with FIFO.

Structural properties of congested equilibria:

Construct the dual graph for every mode μ. If
- the **dual graph is outflow connected**, and
- the inflow vector $\boldsymbol{\lambda_i^\mu > 0}$ for all time,

Then, in mode μ,
- an equilibrium **exists**,
- the equilibrium is **unique**,
- the equilibrium is **locally stable**, with
- a **Lyapunov function** is given by, $V(\rho) = V(\rho) = \rho^T \ln\frac{\rho}{\rho^*} + (\rho^* - \rho)^T 1_N$,
- **region of attraction** can be efficiently bounded by polyhedral bounds.

Further, if there exists an equilibrium ρ_μ^* corresponding to inflow λ in mode μ, then it is **persistent (robust)**, that is, for any ρ_μ^*, if $\|\rho(t_0) - \rho_\mu^*\| < \epsilon$, $\epsilon \in \mathbb{R}$, then, there exists $\Delta \in \mathbb{R}_+$, such that $\|\rho(t) - \rho_\mu^*\| < \epsilon_\mu(\Delta)$, $\epsilon_\mu(\Delta) \in \mathbb{R}$, for all $t \geq t_0$ if $\lambda(t) \in [\lambda - \Delta, \lambda + \Delta]$ for all $t \geq t_0$.

Solution: Transform the system into an equivalent compartmental system that is easier to analyze

Chemical Reaction Network Analogy

CRN	CTM
Molecule	Vehicle
Molecular species/complex	Cell
Concentration	Density
Reaction rates	Velocities

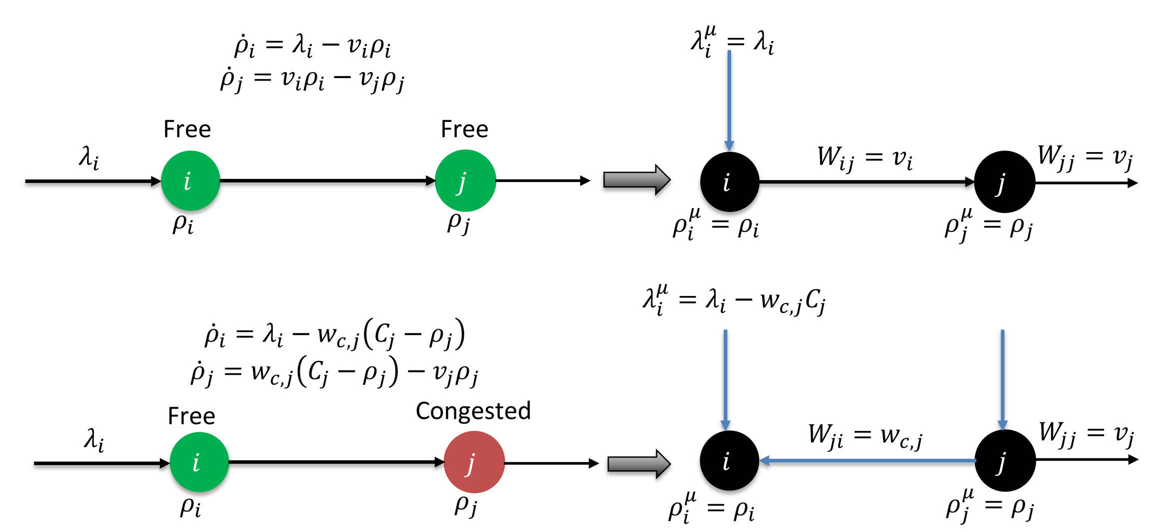
- Convert every mode of the traffic model into an equivalent single species chemical reaction network.
- Dynamical behavior must be preserved.
- The concentration of the reactant and product change as

$$\dot{\rho}_r = -W\rho_r,\ \dot{\rho}_p = W\rho_r$$

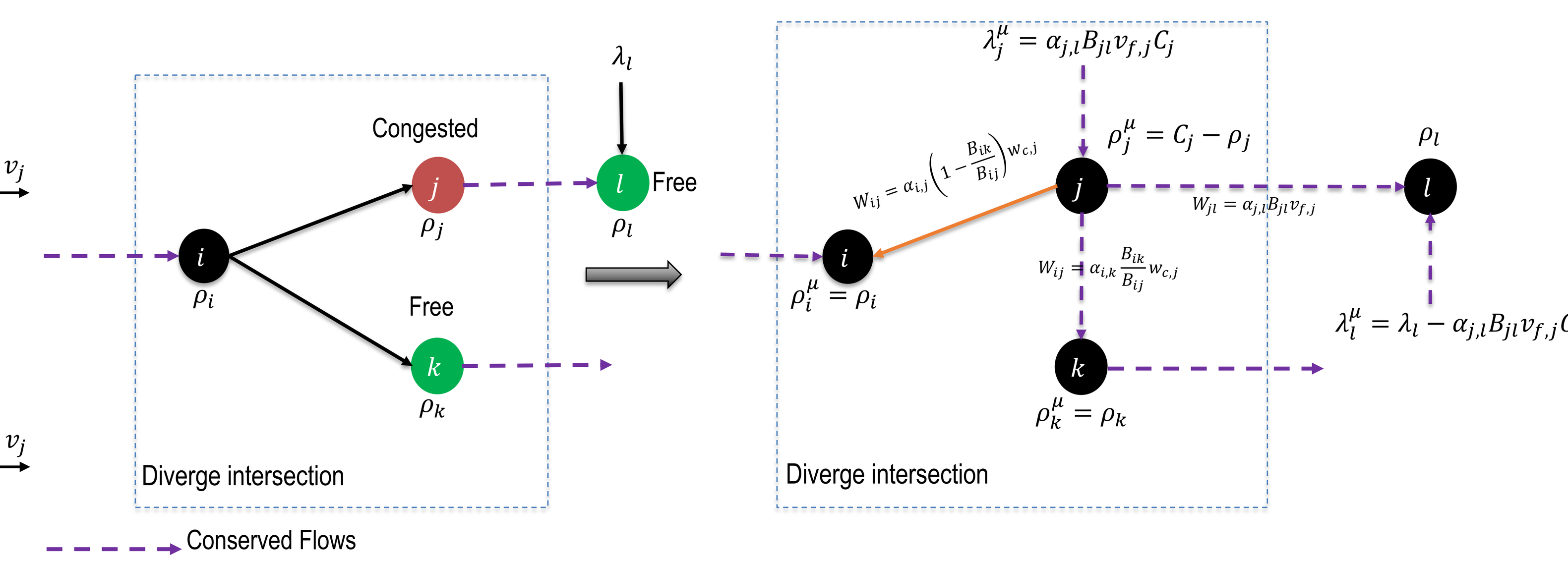
where W is the rate of the reaction.
- For a network,

$$\dot{\rho}_i^\mu(t) = \sum_{j=1}^{N} W_{ji}\rho_j^\mu(t) - \sum_{j=1}^{N} W_{ij}\rho_i^\mu(t) + \lambda_i^\mu(t)$$

Dual Graph:



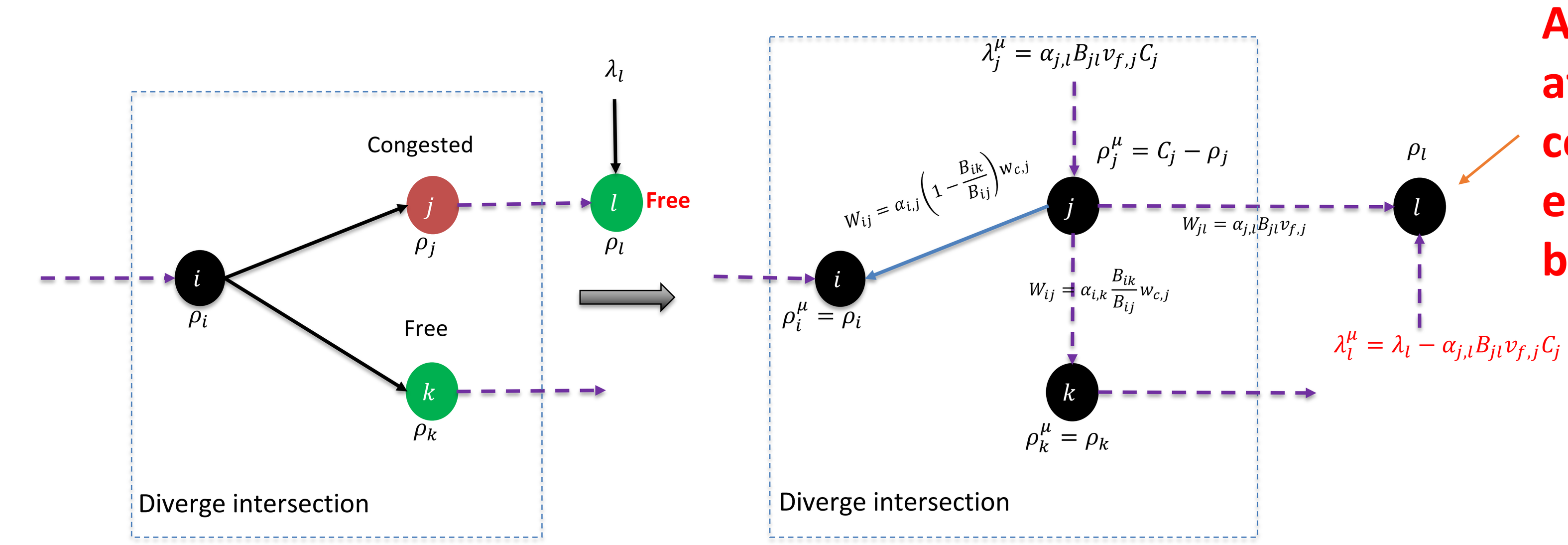
Mass conservation law: Need to express the traffic flow dynamics in this form!



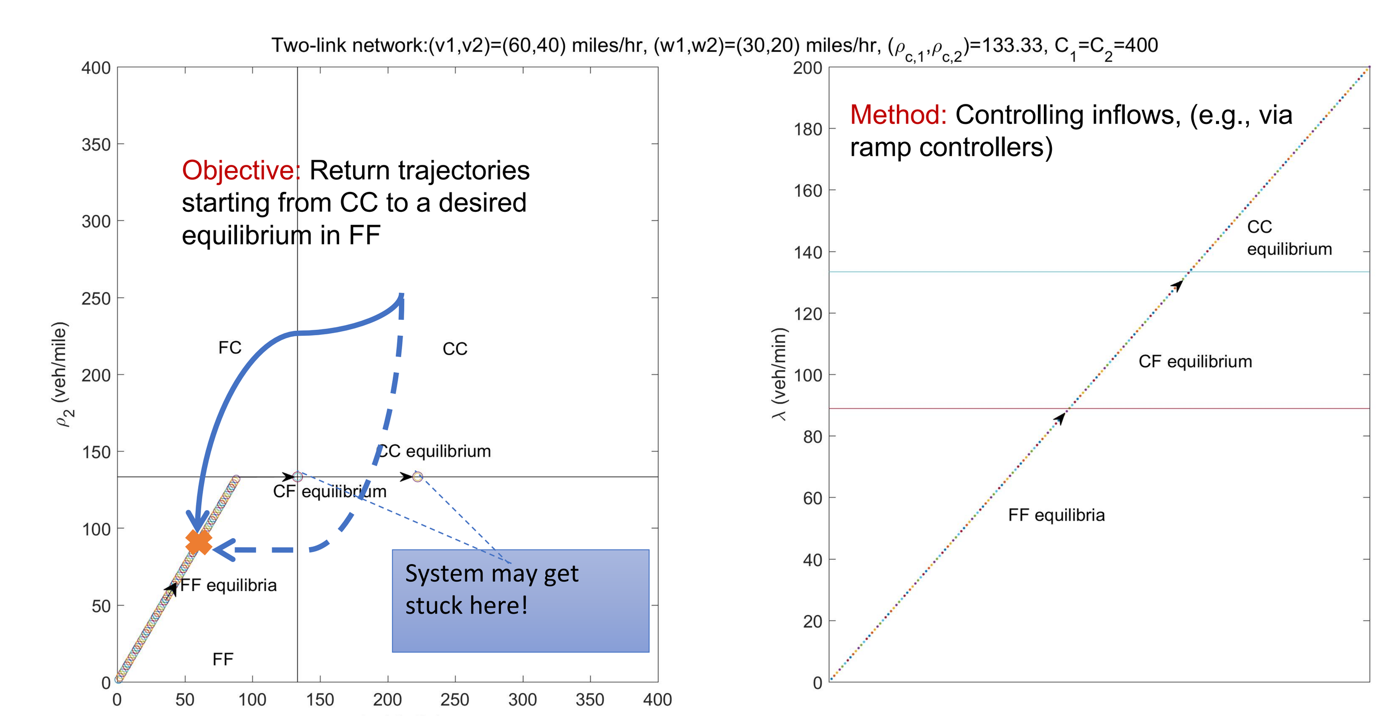
Not all flows are conserved – **relaxation of mass conservation law!**

Application: Identifying and eliminating bottlenecks

- Existence and stability of congested equilibria depends on (i) inflows, (ii) forward and backward velocities, (iii) network structure.
- **Condition $\lambda^\mu > 0$ required for existence of congested equilibria – undesirable congested equilibria can be eliminated by forcing this condition to be violated.**
- **λ^μ depends on system velocities** - velocities can be increased by increasing the capacity of the corresponding cell.
- Therefore, by adding sufficient capacity such that the condition $\lambda^\mu > 0$ is violated in particular modes, certain congested equilibria may be eliminated.
- Location at which capacity must be added can be determined by analyzing dual graph and minimizing the number and region of attraction of congested equilibria.
- Similar analysis for adding new routes or actuator placement.



Adding capacity at this "free" cell will eliminate the bottleneck!



Application: Designing ramp controllers to eliminate congestion