Modelling Options for Networks with Delay: DDEs, DDFs, PDEs, and PIEs

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Control of a Network of Vehicles with Delay

Consider the dynamics of a swarm of UAVs:

$$\dot{x}_i(t) = a_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_j(t - \hat{\tau}_{ij}) + b_{1i} w(t - \bar{\tau}_i) + b_{2i} u(t - h_i)$$

$$z(t) = C_1 x(t) + D_{12} u(t) \qquad \text{Regulated Output}$$

$$y_i(t) = c_{2i} x_i(t - \tilde{\tau}_i) + d_{21i} w(t - \tilde{\tau}_i) \qquad \text{Sensed output}$$

Dynamics:

- a_i is the internal dynamics of UAV i
- a_{ij} is the effect of UAV j on UAV i.
- b_{1i} is the effect of noise on UAV i
- b_{2i} is the effect of the controller on UAV i
- c_{2i} is the measured output of from UAV i
- d_{21i} is the effect of noise on the sensor on UAV i
- C_1 is the output of states to minimize in the optimal control problem
- D_{12} is the actuator output to minimize in the optimal control problem

Delays:

- $\hat{\tau}_{ij}$ is the state delay from UAV j to UAV i
- h_i is the input delay from controller to reach UAV i
- $\bar{\tau}_i$ is the process delay (wind, tracking signal, et c.) for UAV i
- $\tilde{\tau}_i$ is the measurement delay from UAV i to controller

Optimal Control Form using Delay-Diff. Equations (DDEs)

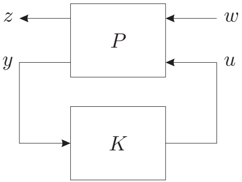
General Form of Optimal Control Problem using DDEs:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + B_2 u(t) + \sum_{i=1}^{N} \left(A_i x(t-\tau_i) + B_{1i} w(t-\tau_i) + B_{2i} u(t-\tau_i) \right)$$

$$z(t) = C_{10} x(t) + D_{11} w(t) + D_{12} u(t) + \sum_{i=1}^{N} \left(C_{1i} x(t-\tau_i) + D_{11i} w(t-\tau_i) + D_{12i} u(t-\tau_i) \right)$$

$$y(t) = C_{20} x(t) + D_{21} w(t) + D_{22} u(t) + \sum_{i=1}^{N} \left(C_{2i} x(t-\tau_i) + D_{21i} w(t-\tau_i) + D_{22i} u(t-\tau_i) \right)$$

Equations: State Equation; Regulated Output; Sensed Output



Problem:

- No network Structure!

Problems:

- Utterly Intractable
- Can't represent some networks

For Simplicity, I leave off the distributed-delay terms.
EVERYTHING is infinite-dimensional - x and w and u

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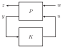

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EVERYTHING is infinite-dimensional - x and w and u

9(N+1) matrices needed to define system

A Network as a DDE

Network Model (neglecting state delays):

$$\dot{x}_i(t) = a_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_j(t) + b_{1i} w(t-\tau_i) + b_{2i} u(t-\tau_{N+i})$$
$$z(t) = C_1 x(t) + D_{12} u(t)$$
$$y_i(t) = c_{2i} x_i(t-\tau_{2N+i}) + d_{21i} w(t-\tau_{2N+i}).$$

DDE Representation:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} B_{1i} w(t-\tau_i) + \sum_{j=N+1}^{2N} B_{2i} u(t-\tau_i)$$
$$z(t) = C_{10} x(t) + D_{12} u(t)$$
$$y(t) = \sum_{i=2N+1}^{3N} C_{2i} x(t-\tau_i) + \sum_{i=2N+1}^{3N} D_{21i} w(t-\tau_i)$$

Problem: delayed information has dimension $3N(n_x \cdot N + n_w + n_u)$ where here N is # of UAVs.

Some Networks CANNOT be modelled using DDEs

Network model with input delay:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + \sum_{i=1}^{N} B_{2i} u(t - \tau_i)$$
$$z(t) = C_1 x(t) + D_{12} u(t)$$
$$y(t) = C_2 x(t) + D_{21} w(t) + \sum_{i=1}^{N} D_{22i} u(t - \tau_i).$$

with STATIC FEEDBACK:

$$u(t) = F y(t)$$

Now, substituting $u(t) = F y(t)$ into the sensed output term, we obtain solutions of the form

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + \sum_{i} B_{2i} F y(t - \tau_i)$$
$$z(t) = C_1 x(t) + D_{12} F y(t)$$
$$y(t) = C_2 x(t) + D_{21} w(t) + \sum_{i=1}^{N} D_{22i} F y(t - \tau_i). \tag{1}$$

There is no DDE which satisfies Eqns. (1) due to the recursion in the output.

Advantages/Disadvantages of DDE formulation

Advantages:

- Well studied
  - State delay well-studied using LK functions
  - Input delay handled by Smith predictors
  - Padé approximations, LMI methods, SOS methods, etc.
- Always Well-Posed

Disadvantages?

- Lots of delay terms everywhere
- Implies lots of information is delayed
- Can't represent some models
- Many tools implicitly treat as a PDE

Optimal Control via Diff.-DiFFerence Equations (DDFs)

DDFs separate delayed information into low-dimensional channels

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + B_1 w(t) + B_2 u(t) + B_v v(t)\\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) + D_{1v} v(t)\\
y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t) + D_{2v} v(t)\\
r_i(t) &= C_{ri} x(t) + B_{r1i} w(t) + B_{r2i} u(t) + D_{rvi} v(t)
\end{aligned}$$

$$v(t) = \sum_{i=1}^{N} C_{vi} r_i(t-\tau_i) + \sum_{i=1}^{N} \int_{-\tau_i}^{0} C_{vdi}(s) r_i(t+s) ds$$

The information which is being delayed is stored in the information channels $r_i(t)$

- State in green is the infinite-dimensional part of the state
- Allows for lower-dimensional states
- Allows for simple difference equations (Discrete time) using D_{rvi}

Converting a DDE to a DDF

DDE Formulation:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + B_2 u(t) + \sum_{i=1}^{N} \Big(A_i x(t-\tau_i) + B_{1i} w(t-\tau_i) + B_{2i} u(t-\tau_i) \Big)$$

$$z(t) = C_{10} x(t) + D_{11} w(t) + D_{12} u(t) + \sum_{i=1}^{N} \Big(C_{1i} x(t-\tau_i) + D_{11i} w(t-\tau_i) + D_{12i} u(t-\tau_i) \Big)$$

$$y(t) = C_{20} x(t) + D_{21} w(t) + D_{22} u(t) + \sum_{i=1}^{N} \Big(C_{2i} x(t-\tau_i) + D_{21i} w(t-\tau_i) + D_{22i} u(t-\tau_i) \Big)$$

DDF Formulation:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + B_2 u(t) + B_v v(t)$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) + D_{1v} v(t)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) + D_{2v} v(t)$$

$$r_i(t) = C_{ri} x(t) + B_{r1i} w(t) + B_{r2i} u(t) + D_{rvi} v(t)$$

$$v(t) = \sum_{i=1}^{N} C_{vi} r_i(t-\tau_i) + \sum_{i=1}^{N} \int_{-\tau_i}^{0} C_{vdi}(s) r_i(t+s) ds$$

Converting a DDE to a DDF

DDF Formulation:

$$\begin{aligned}
\dot{x}(t) &= A_0 x(t) + B_1 w(t) + B_2 u(t) + B_v v(t)\\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) + D_{1v} v(t)\\
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r_i(t) &= C_{ri} x(t) + B_{r1i} w(t) + B_{r2i} u(t) + D_{rvi} v(t)
\end{aligned}$$

$$v(t) = \sum_{i=1}^{K} C_{vi} r_i(t-\tau_i) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} C_{vdi}(s) r_i(t+s) ds$$

In order of appearance (all other matrices unchanged):

$$B_v = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \qquad D_{1v} = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \qquad D_{2v} = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$$

$$C_{ri} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \qquad B_{r1i} = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \qquad B_{r2i} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \qquad D_{rvi} = 0$$

$$C_{vi} = \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & D_{22i} \end{bmatrix}, \qquad C_{vdi}(s) = \begin{bmatrix} A_{di}(s) & B_{1di}(s) & B_{2di}(s) \\ C_{1di}(s) & D_{11di}(s) & D_{12di}(s) \\ C_{2di}(s) & D_{21di}(s) & D_{22di}(s) \end{bmatrix}$$

Reverse Transformation (DDF to DDE) Not Possible

Standard Network Model using DDF Formulation

Network Model (neglecting state delays):

$$\dot{x}_i(t) = a_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_j(t) + b_{1i} w(t-\tau_i) + b_{2i} u(t-\tau_{N+i})$$
$$z(t) = C_1 x(t) + D_{12} u(t)$$
$$y_i(t) = c_{2i} x_i(t-\tau_{2N+i}) + d_{21i} w(t-\tau_{2N+i}).$$

The DDF representation:

$$\dot{x}(t) = A_0 x(t) + \sum\nolimits_{i=1}^{2N} v_i(t)$$
$$z(t) = C_{10} x(t) + D_{12} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} v(t)$$

$$v(t) = \begin{bmatrix} r_1(t-\tau_1) \\ \vdots \\ r_{3N}(t-\tau_{3N}) \end{bmatrix} r_i(t) = \begin{cases} b_{1i} w(t) & i \in [1, N] \\ b_{2,i-N} u(t) & i \in [N+1, 2N] \\ c_{2,i-2N} x_{i-2N}(t) + d_{21,i-2N} w(t) & i \in [2N+1, 3N]. \end{cases}$$

The state-space dimension of the delayed component is $(2n_x + n_y)N$ (vs. $3N(n_x N + n_w + n_u)$ for the DDE). $y_i \in \mathbb{R}^{n_y}$, $x_i \in \mathbb{R}^{n_x}$.

Standard Network Model using DDF Formulation

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$$\dot{x}_i(t) = a_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_j(t) + b_{1i} w(t-\tau_i) + b_{2i} u(t-\tau_{N+i})$$

$$z(t) = C_1 x(t) + D_{12} u(t)$$

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The DDF representation:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2N} v_i(t)$$

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The state-space dimension of the delayed component is $(2n_x + n_y)N$ (vs. $3N(n_x N + n_w + n_u)$ for the DDE). $y_i \in \mathbb{R}^{n_y}$, $x_i \in \mathbb{R}^{n_x}$.

Definition of r was chosen assuming dimension of x_i is less than that of w or u. Otherwise choose

$$r_i(t) = \begin{cases} w(t) & i \in [1,N] \\ u(t) & i \in [N+1,2N] \\ \begin{bmatrix} x_{i-2N}(t) \\ w(t) \end{bmatrix} & i \in [2N+1,3N]. \end{cases}$$

or

$$r_i(t) = \begin{cases} w(t) & i \in [1,N] \\ u(t) & i \in [N+1,2N] \\ c_{2,i-2N} x_{i-2N}(t) + d_{21,i-2N} w(t) & i \in [2N+1,3N]. \end{cases}$$

A Network which is a DDF but not a DDE

Static State Feedback Model:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + \sum_i B_{2i} F y(t-\tau_i)$$

$$z(t) = C_1 x(t) + D_{12} F y(t)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + \sum_{i=1}^{N} D_{22i} F y(t-\tau_i).$$

DDF Representation:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + \sum\nolimits_{i=1}^{N} B_{2i} v_i(t)$$

$$z(t) = (C_1 + D_{12} F C_2) x(t) + D_{12} F D_{21} w(t) + D_{12} F D_{2v} \sum\nolimits_{i=1}^{N} D_{22i} v_i(t)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + \sum\nolimits_{i=1}^{N} D_{22i} v_i(t)$$

$$r_i(t) = F C_2 x(t) + F D_{21} w(t) + F D_{22i} v_i(t)$$

$$v_i(t) = r_i(t-\tau_i)$$

Advantages/Disadvantages of DDF formulation

Advantages:

- Use of low dimensional channels
- Reduces computation complexity of all analysis and control algorithms
  - Padé approximations, LMI methods, SOS methods, etc.
- Can represent difference equations

Disadvantages:

- Relatively few analysis and controls techniques available
  - Literature is Sparse
- Well-posedness is not assumed

Optimal Control via ODE-PDE Formulation

Almost identical to DDF formulation

- Information channels are represented by PDEs of form $u_t = u_s$

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + B_2 u(t) + B_v v(t)$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) + D_{1v} v(t)$$
$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) + D_{2v} v(t)$$
$$\dot{\phi}_i(t,s) = \frac{1}{\tau_i}\phi_{i,s}(t,s) \qquad \phi_i(t,0) = C_{ri} x(t) + B_{r1i} w(t) + B_{r2i} u(t) + D_{rvi} v(t)$$
$$v(t) = \sum_{i=1}^{N} C_{vi}\phi_i(t,-1) + \sum_{i=1}^{N} \int_{-1}^{0} \tau_i C_{vdi}(\tau_i s)\phi_i(t,s) ds$$

An extension of the DDF formulation

- $\phi_i(t)$ is same as $r_i(t)$ was in the DDF
- PDE state is in green.
- Coupled to ODE through Boundary Conditions.

Advantages/Disadvantages of ODE-PDE formulation

Class of Systems Considered: Includes the DDF class of systems

Advantages:

- Use of low dimensional channels
- Tools developed for PDEs can be applied
  - Discretization schemes
  - Backstepping methods for control
- More physical interpretation?

Disadvantages:

- Use of unbounded operators
  - Dirac operators
  - Differential operators

The Partial-Integral Equation (PIE) Formulation

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{B}_{T_1}\dot{w}(t) + \mathcal{B}_{T_2}\dot{u}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11}w(t) + \mathcal{D}_{12}u(t),\\ y(t) &= \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t) + \mathcal{D}_{22}u(t),\end{aligned} \tag{2}$$

where $\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}$ are Partial Integral (4-PI) operators of the form

$$\left(\mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{smallmatrix}\right\} \underbrace{\begin{bmatrix} x \\ \mathbf{\Phi}\end{bmatrix}}_{\mathbf{x}}\right)(s) := \begin{bmatrix} Px + \int_{-1}^{0} Q_1(s)\mathbf{\Phi}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s)\end{bmatrix}$$

where $\mathcal{P}_{\{R_i\}}$ is a 3-PI operator of the form:

$$\left(\mathcal{P}_{\{R_i\}}\mathbf{\Phi}\right)(s) := R_0(s)\mathbf{\Phi}(s) + \int_{-1}^{s} R_1(s,\theta)\mathbf{\Phi}(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\mathbf{\Phi}(\theta)d\theta.$$

The state is in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

- Dimensions are the same as the ODE-PDE framework
- 4-PI operators are bounded, algebraic.
- Can be used as matrices in LMIs (yielding LOIs)

Linear Operator Inequalities (LOIs) in the PIE Formulation

PIE formulation of System:

All results on this page are for no input delays ($B_{T1} = 0$) and no process delays ($B_{T2} = 0$)

- Extension OK to input delays.

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t)\\ z(t) &= \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t),\\ y(t) &= \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21} w(t) + \mathcal{D}_{22} u(t).\end{aligned}$$

KYP and H_∞-Gain

If there exists $\mathcal{P} = \mathcal{P}\left\{\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{smallmatrix}\right\} \geq 0$ such that

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11}^* & \mathcal{B}_1^*\mathcal{P}\mathcal{T} \\ \mathcal{D}_1 & -\gamma I & \mathcal{C}_1 \\ \mathcal{T}^*\mathcal{P}\mathcal{B}_1 & \mathcal{C}_1^* & \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}\end{bmatrix} < 0$$

then $\|z\|_{L_2} \leq \gamma\|\omega\|_{L_2}$.

H_∞-Optimal Full State Feedback:

If there exist $\mathcal{P} = \mathcal{P}\left\{\begin{smallmatrix} P, & Q \\ Q^T, & \{R_i\}\end{smallmatrix}\right\} > 0$ and $\mathcal{Z} = \mathcal{P}\left\{\begin{smallmatrix} Z_1, & Z_2 \\ \emptyset, & \{\emptyset\}\end{smallmatrix}\right\}$ such that

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}_1\mathcal{P} + \mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ *^* & -\gamma I & \mathcal{B}_1^* \\ *^* & *^* & (\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^*\end{bmatrix} < 0$$

then if $u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)$, $\|z\|_{L_2} \leq \gamma\|\omega\|_{L_2}$.

H_∞-Optimal Estimator Design:

If there exist $\mathcal{P} = \mathcal{P}\left\{\begin{smallmatrix} P, & Q \\ Q^T, & \{R_i\}\end{smallmatrix}\right\} \geq 0$ and $\mathcal{Z} = \mathcal{P}\left\{\begin{smallmatrix} Z_1, & \emptyset \\ Z_2, & \{\emptyset\}\end{smallmatrix}\right\}$ such that

$$\begin{bmatrix} -\gamma I & -\mathcal{D}_{11}^* & -(\mathcal{P}\mathcal{B}_1 + \mathcal{Z}\mathcal{D}_{21})^*\mathcal{T} \\ *^* & -\gamma I & \mathcal{C}_1 \\ *^* & *^* & (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)^*\mathcal{T} + \mathcal{T}^*(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\end{bmatrix} < 0$$

then if $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$, $\|\hat{z} - z\|_{L_2} \leq \gamma\|\omega\|_{L_2}$ where

$$\begin{aligned}\mathcal{T}\dot{\hat{\mathbf{x}}}(t) &= \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}(\hat{y}(t) - y(t))\\ \hat{y}(t) &= \mathcal{C}_2\hat{\mathbf{x}}(t) \quad \hat{z}(t) = \mathcal{C}_1\hat{\mathbf{x}}(t)\end{aligned}$$

PIETOOLS Code for solving Linear Operator Inequalities

PIE formulation of System:

```
pvar s,th,gam;
T = sosprogram([s,th],gam);
opvar A,B1,B2,C1,C2,D11,D12,D21,E;
A=..;B1=..;B2=..;C1=..;C2=..;
D11=..;D12=..;D21=..;E=..;
```

KYP and H_∞-Gain

```
[T,P] = sos_posopvar(T,dim,I,s,th);
D = [-gam*I   D11'      B1'*P*E;
     D11        -gam*I   C1;
     E'*P*B1  C1'       A'*P*E+E'*P*A];
T = sosopineq(T,D);
T = sossetobj(T,gam);
```

H_∞-Optimal Full State Feedback:

```
[T,P] = sos_posopvar(T,dim,I,s,th,deg);
[T,P] = sos_opvar(T,dim,I,s,th,deg);
M33 = (A*P+B2*Z)*E'+E*(A*P+B2*Z)'
M13 = (C1*P+D12*Z)*E'
D = [-gam*I   D11'      M13;
       D11    -gam*I     B1';
       M13'    B1       M33];
T = sosopineq(T,D);
T = sossetobj(T,gam);
T = sossolve(T);
```

H_∞-Optimal Estimator Design:

```
[T,P] = sos_posopvar(T,dim,I,s,th);
[T,P] = sos_opvar(T,dim,I,s,th,deg);
M33 = (P*A+Z*C2)'*E+E'*(P*A+Z*C2)
M13 = -B1'*P*E-D21'*Z'*E
D = [-gam*I   D11'      M13;
       D11    -gam*I     C1;
       M13'    C1'       M33];
T = sosopineq(T,D);
T = sossetobj(T,gam);
T = sossolve(T);
```

Advantages/Disadvantages of PIE formulation

Class of Systems Considered: Includes the DDF class of systems

Advantages:

- Actually Old (Barbasin type)
- Use of low dimensional channels
- Tools developed for ODEs can be applied
  - LMIs
  - Manipulation is easy

Disadvantages:

- Not many results in this area
  - Relatively New
  - Literature is Sparse
- Similar Structure to Singular Systems

A network control problem in the DDE formulation

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t), \quad y(t) = Cx(t) + D_1 w(t) + D_2 u(t)$$

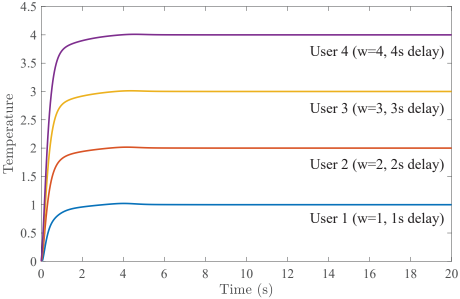
where

$$A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B_1 = \begin{bmatrix} -I \\ -\hat{\Gamma} + \mathrm{diag}(\alpha_1 \dots \alpha_K) \end{bmatrix}$$

$$\hat{A}_i(:,i) = \alpha_i \begin{bmatrix} \gamma_{i,1} & \dots & \gamma_{i,i-1} & -1 & \gamma_{i,i-1} & \dots & \gamma_{i,K} \end{bmatrix}^T$$

$$\hat{\Gamma}_{ij} = \alpha_j \gamma_{ij} = \begin{bmatrix} q_1 & \dots & q_K \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0 \\ .1I \end{bmatrix}$$



Complexity: 8 states, 4 delays, 4 inputs, 4 disturbances, 8 regulated outputs

Results: A Matlab simulation of the step response of the closed-loop dynamics ($T_{2i}(t)$) with 4 users (w_i and τ_i as indicated) coupled with the controller with closed-loop gain of $.48$

DDF Formulation:

$$\dot{x}(t) = A_0 x(t) + B_1 w(t) + B_2 u(t) + B_v v(t)$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) + D_{1v} v(t)$$
$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) + D_{2v} v(t)$$
$$r_i(t) = C_{ri} x(t) + B_{r1i} w(t) + B_{r2i} u(t) + D_{rvi} v(t)$$
$$v(t) = \sum_{i=1}^{K} C_{vi} r_i(t - \tau_i) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} C_{vdi}(s) r_i(t+s) ds$$

PIE Formulation:

$$\begin{aligned} \mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{B}_{T_1}\dot{w}(t) + \mathcal{B}_{T_2}\dot{u}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t) \\ z(t) &= \mathcal{C}_1 \mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \\ y(t) &= \mathcal{C}_2 \mathbf{x}(t) + \mathcal{D}_{21} w(t) + \mathcal{D}_{22} u(t), \end{aligned} \tag{3}$$

Converting a DDF to a PIE

$$\mathcal{A} := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{A}_0, & \mathbf{A} \\ 0, & \{I_\tau, 0, 0\}\end{smallmatrix}\right\}, \qquad \mathcal{T} := \mathcal{P}\left\{\begin{smallmatrix} I, & 0 \\ \mathbf{T}_0, & \{0, \mathbf{T}_a, \mathbf{T}_b\}\end{smallmatrix}\right\}$$

$$\mathcal{B}_1 := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{B}_1, & \emptyset \\ 0, & \{\emptyset\}\end{smallmatrix}\right\}, \qquad \mathcal{B}_2 := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{B}_2, & \emptyset \\ 0, & \{\emptyset\}\end{smallmatrix}\right\}, \qquad \mathcal{B}_{T_1} := \mathcal{P}\left\{\begin{smallmatrix} 0, & \emptyset \\ \mathbf{T}_1, & \{\emptyset\}\end{smallmatrix}\right\}, \qquad \mathcal{B}_{T_2} := \mathcal{P}\left\{\begin{smallmatrix} 0, & \emptyset \\ \mathbf{T}_2, & \{\emptyset\}\end{smallmatrix}\right\}$$

$$\mathcal{C}_1 := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{C}_{10}, & \mathbf{C}_{11} \\ \emptyset, & \{\emptyset\}\end{smallmatrix}\right\}, \quad \mathcal{C}_2 := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{C}_{20}, & \mathbf{C}_{21} \\ \emptyset, & \{\emptyset\}\end{smallmatrix}\right\}, \quad \mathcal{D}_{ij} := \mathcal{P}\left\{\begin{smallmatrix} \mathbf{D}_{ij}, & \emptyset \\ \emptyset, & \{\emptyset\}\end{smallmatrix}\right\}$$

$$\mathbf{T}_0 = \begin{bmatrix} C_{r1} + D_{rv1}C_{vx} \\ \vdots \\ C_{rK} + D_{rvK}C_{vx} \end{bmatrix}, \qquad \mathbf{T}_1 = \begin{bmatrix} B_{r11} + D_{rv1}D_{vw} \\ \vdots \\ B_{r1K} + D_{rvK}D_{vw} \end{bmatrix}, \qquad \mathbf{T}_2 = \begin{bmatrix} B_{r21} + D_{rv1}D_{vu} \\ \vdots \\ B_{r2K} + D_{rvK}D_{vu} \end{bmatrix}$$

$$\mathbf{T}_a(s) = \begin{bmatrix} D_{rv1}\begin{bmatrix} C_{I1}(s) & \cdots & C_{IK}(s)\end{bmatrix} \\ \vdots \\ D_{rvK}\begin{bmatrix} C_{I1}(s) & \cdots & C_{IK}(s)\end{bmatrix} \end{bmatrix}, \qquad \mathbf{T}_b = -I + \mathbf{T}_a(s), \quad I_\tau = \begin{bmatrix} \frac{1}{\tau_1}I & & \\ & \ddots & \\ & & \frac{1}{\tau_K}I \end{bmatrix},$$

$$\mathbf{A}_0 = A_0 + B_vC_{vx}, \; \mathbf{A}(s) = B_v\begin{bmatrix} C_{I1}(s) & \cdots & C_{IK}(s)\end{bmatrix}, \; \mathbf{B}_1 = B_1 + B_vD_{vw}, \; \mathbf{B}_2 = B_2 + B_vD_{vu},$$

$$\mathbf{C}_{10} = C_1 + D_{1v}C_{vx}, \qquad \mathbf{C}_{11} = D_{1v}\begin{bmatrix} C_{I1}(s) & \cdots & C_{IK}(s)\end{bmatrix},$$

$$\mathbf{C}_{20} = C_2 + D_{2v}C_{vx}, \qquad \mathbf{C}_{21} = D_{2v}\begin{bmatrix} C_{I1}(s) & \cdots & C_{IK}(s)\end{bmatrix},$$

$$\mathbf{D}_{11} = (D_{11} + D_{1v}D_{vw}), \; \mathbf{D}_{12} = (D_{12} + D_{1v}D_{vu}), \; \mathbf{D}_{21} = (D_{21} + D_{2v}D_{vw}), \; \mathbf{D}_{22} = (D_{22} + D_{2v}D_{vu})$$

$$\hat{C}_{vi} = C_{vi} + \int_{-1}^{0} \tau_i C_{vdi}(\tau_i s)ds, \qquad D_I = \left(I - \left(\sum_{i=1}^{K} \hat{C}_{vi}D_{rvi}\right)\right)^{-1}, \quad C_{vx} = D_I\left(\sum_{i=1}^{K} \hat{C}_{vi}C_{ri}\right)$$

$$D_{vw} = D_I\left(\sum_{i=1}^{K} \hat{C}_{vi}B_{r1i}\right), \; D_{vu} = D_I\left(\sum_{i=1}^{K} \hat{C}_{vi}B_{r2i}\right), \; C_{Ii}(s) = -D_I\left(C_{vi} + \tau_i\int_{-1}^{s} C_{vdi}(\tau_i\eta)d\eta\right)$$

Conclusion: What is the best way to represent a network?

The Last Slide (Thanks to NSF CNS-1739990)

Depends on your goal

- Control? Exploration? Consensus?

DDEs:

- Convenient for small or delay-free networks
- Can use Padé approximation
- Smith Predictors, SOS
- Lots of literature
- No Difference equations

DDFs:

- Can use Padé approximation
- Not much literature
- Can be used for large networks and lots of delays
- Can combine discrete/continuous time
- OK for static feedback

ODE-PDEs:

- All the advantages of DDFs
- Good for intuition
- Good if you know how to use backstepping
- Can apply PDE discretization schemes

PIEs:

- Can use intuition from ODEs
- Can use PIETOOLS
- good for optimal control
- MAY be good for simulation (no BC's)

Discussion?

More Options?

Illustration of H_∞ Gain Analysis

Example 1:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} -.5 \\ 1 \end{bmatrix} w(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

d	1	2	3	Padé	[Fridman 2001]	[Shaked 1998]
$\gamma_{\min}$	.2373	.2365	.2365	.2364	.32	2

Example 2: Stable for $\tau \in [.100173, 1.71785]$:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

We plot bounds for the H_∞ norm as the delay varies within this interval. As expected, the H_∞ norm approaches infinity quickly as we approach the limits of the stable region.

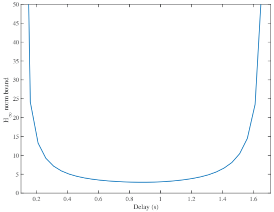


Figure: Calculated H_∞ norm bound vs. delay for Ex. 2

The Inverse of a 4-PI Operator is a 4-PI Operator!

Result from Keqin Gu

How to find (Note $R_1 = R_2$)

$$\mathcal{K} = \mathcal{P}\left\{\begin{smallmatrix} Z_1, & Z_2 \\ \emptyset, & \{\emptyset\} \end{smallmatrix}\right\}\mathcal{P}\left\{\begin{smallmatrix} P, & Q \\ Q^T, & \{S,\, R,\, R\} \end{smallmatrix}\right\}^{-1}?$$

Assume Q and R are polynomial

Extract Polynomial Coefficients: $Q(s) = HZ(s)$ and $R(s,\theta) = Z(s)^T \Gamma Z(\theta)$.

Then $\mathcal{P}\left\{\begin{smallmatrix} P, & Q \\ Q^T, & \{S,\, R,\, R\} \end{smallmatrix}\right\}^{-1} = \mathcal{P}\left\{\begin{smallmatrix} \hat{P}, & \hat{Q} \\ \hat{Q}^T, & \{\hat{S},\, \hat{R},\, \hat{R}\} \end{smallmatrix}\right\}$ where

$$\hat{P} = \left(I - \hat{H} V H^T\right) P^{-1}, \qquad \hat{Q}(s) = \frac{1}{\tau}\hat{H} Z(s) S(s)^{-1}$$

$$\hat{S}(s) = \frac{1}{\tau^2} S(s)^{-1} \qquad \hat{R}(s,\theta) = \frac{1}{\tau} S(s)^{-1} Z(s)^T \hat{\Gamma} Z(\theta) S(\theta)^{-1},$$

where

$$\hat{H} = P^{-1} H \left(V H^T P^{-1} H - I - V\Gamma\right)^{-1}$$

$$\hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + V\Gamma)^{-1},$$

$$V = \int_{-\tau}^{0} Z(s) S(s)^{-1} Z(s)^T ds$$

Boring Numerical Controller Synthesis Examples

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

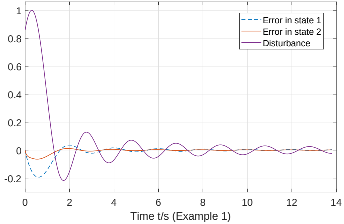
$$y(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

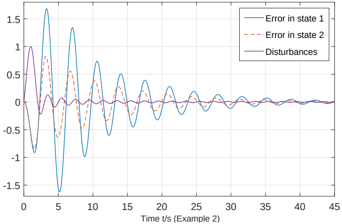
d	1	2	3	Padé	Fridman 2003	Li 1997
$\gamma_{\min}(\tau = .999)$	.10001	.10001	.10001	.1000	.22844	1.8822
$\gamma_{\min}(\tau = 2)$	1.43	1.36	1.341	1.340	∞	∞
CPU sec	.478	.879	2.48	2.78	N/A	N/A

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} -.5 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

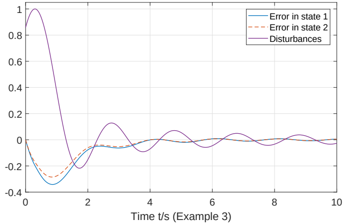
$$y(t) = \begin{bmatrix} 1 & -.5 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

d	1	2	3	Padé
$\gamma_{\min}(\tau = .3)$	.3953	.3953	.3953	.3953
CPU sec	.655	1.248	2.72	2.91





γ_{min}	Example 1			Example 2			Example 3		
	d=1	d=2	d=4	d=1	d=2	d=4	d=1	d=2	d=4
using simplified estimator	0.2371	0.23651	0.23608	7.2111			0.2264		
using generalized estimator	0.2357			7.2111			0.2264		
Padé	0.2357			7.2107			0.2264		



Systems with Input Delays (Control at the Boundary)

- When there is only one input delay, we may alternatively design an estimator using a delayed output (which then becomes a predictor) and is stable in closed-loop using the separation principle
- Control at the boundary is slightly more complex than in-domain control.

$$\mathbf{x}_p = \mathcal{T}\mathbf{x}_f + \mathcal{B}_{T2}u(t) \qquad u(t) = \mathcal{K}\mathbf{x}_f(t)$$

Replace $\mathcal{T} \to \mathcal{T} + \mathcal{B}_{T2}\mathcal{K}$

$$\begin{bmatrix} -\gamma I & \mathcal{D}_1 & (\mathcal{C}\mathcal{P} + \mathcal{D}_2\mathcal{Z})(\mathcal{T} + \mathcal{B}_{T2}\mathcal{K})^* \\ \mathcal{D}_1^T & -\gamma I & \mathcal{B}_1^* \\ (\mathcal{T} + \mathcal{B}_{T2}\mathcal{K})(\mathcal{C}\mathcal{P} + \mathcal{D}_2\mathcal{Z})^* & \mathcal{B}_1 & (\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})(\mathcal{T} + \mathcal{B}_{T2}\mathcal{K})^* + (\mathcal{T} + \mathcal{B}_{T2}\mathcal{K})(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} < 0$$