Optimal Resource Allocation to Control Spreading Processes in Arbitrary Networks

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Text

Project Summary

In a network with a spreading (cascading) process, what's the optimal strategy to allocate scarce protective and corrective control resources?

- Agents in the network are interacting and influencing one another's vulnerability.
- While centrality measures of the agents can be a guide, based on a convex framework, we find a nontrivial resource allocation scheme.
- Our framework is for strongly connected networks of arbitrary structure, including directed networks.

Introduction

Agents in the network are in either susceptible or infected state.

- Protective resources lower infection rate β_i, corrective resources increase recovery rate δ_i.
  Probability of infection evolves according to

$$\frac{dp_i(t)}{dt} = (1 - p_i(t))\,\beta_i \sum_{j=1}^{n} a_{ij} p_j(t) - \delta_i p_i(t). \quad (1)$$

- Control of epidemic outbreak – if the eigenvalue with largest real part of $BA_{\mathcal{G}} - D$ satisfies

$$\Re\left[\lambda_1\left(BA_{\mathcal{G}} - D\right)\right] \le -\varepsilon,$$

for some $\varepsilon > 0$, the disease-free equilibrium ($\boldsymbol{p}^* = \mathbf{0}$) is globally exponentially stable.

Our Model & Problem

Find optimal protective and corrective resource allocation scheme that minimizes the cost and satisfies specified decay rate

$$\underset{\{\beta_i \delta_i\}_{i=1}^n}{\text{minimize}} \ \sum_{i=1}^{n} f_i(\beta_i) + g_i(\delta_i) \quad (2)$$

$$\text{subject to } \Re\left[\lambda_1\left(\text{diag}(\beta_i) A_{\mathcal{G}} - \text{diag}(\delta_i)\right)\right] \le -\varepsilon, \quad (3)$$

$$\underline{\beta}_i \le \beta_i \le \overline{\beta}_i, \quad (4)$$

$$\underline{\delta}_i \le \delta_i \le \overline{\delta}_i, \ i = 1, \ldots, n, \quad (5)$$

where (2) is the total investment, (3) constrains the decay rate to ε, and (4)-(5) maintain the infection and recovery rates in their feasible limits.

Budget Constrained

$$\underset{\varepsilon, \{\beta_i \delta_i\}_{i=1}^n}{\text{maximize}} \ \varepsilon \quad (6)$$

$$\text{subject to } \Re\left[\lambda_1\left(\text{diag}(\beta_i) A_{\mathcal{G}} - \text{diag}(\delta_i)\right)\right] \le -\varepsilon, \quad (7)$$

$$\sum_{i=1}^{n} f_i(\beta_i) + g_i(\delta_i) \le C, \quad (8)$$

$$\underline{\beta}_i \le \beta_i \le \overline{\beta}_i, \quad (9)$$

$$\underline{\delta}_i \le \delta_i \le \overline{\delta}_i, \ i = 1, \ldots, n, \quad (10)$$

Network Interactions

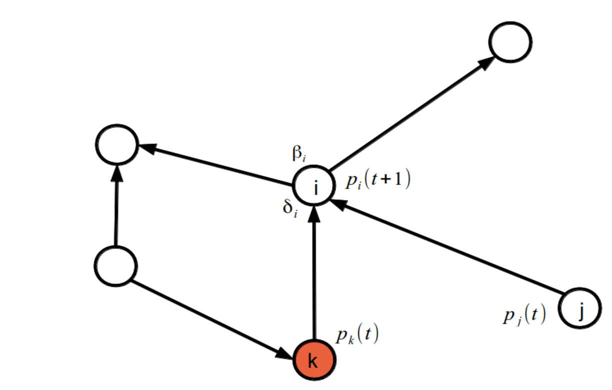

Figure 1: A Network illustrating spread of infection

Cost Functions

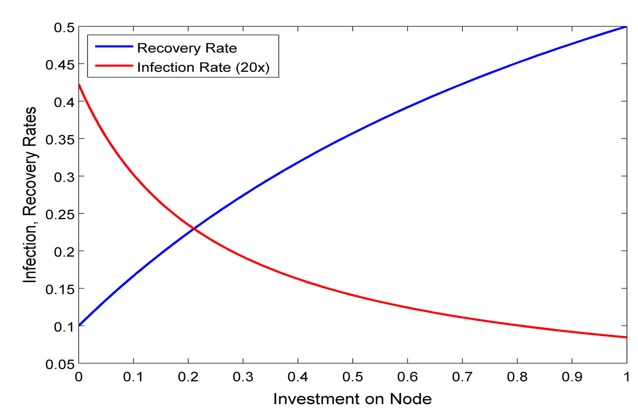

Figure 2: Cost functions for infection and recovery rates

Convex Characterization

Let $M(\mathbf{x})$ comprise posynomials such that $\mathbf{x} \in \mathbb{R}^k_{++}$ and $f_i(\mathbf{x}) \le 1$, where f_i are posynomials. Then, we can minimize $\lambda_1(M(\mathbf{x}))$ by solving the following Geometric Program

$$\underset{\lambda, \{u_i\}_{i=1}^n, \mathbf{x}}{\text{minimize}} \ \lambda \quad (11)$$

$$\text{subject to } \frac{\sum_{j=1}^{n} M_{ij}(\mathbf{x})\, u_j}{\lambda u_i} \le 1, \ i = 1, \ldots, n, \quad (12)$$

$$f_i(\mathbf{x}) \le 1, \ i = 1, \ldots, m. \quad (13)$$

Given a strongly connected graph with posynomial costs, optimal resource allocation for node v_i to solve (6)–(10) are $f_i(\beta_i^*)$ and $g_i(\bar{\Delta} + 1 - \bar{\delta}_i^*)$, where $\bar{\Delta} \triangleq \max\{\delta_i\}_{i=1}^n$ and $\beta_i^*, \bar{\delta}_i^*$ are the optimal solution for β_i and δ_i in

$$\underset{\{u_i, \beta_i, \bar{\delta}_i, t_i\}_{i=1}^n}{\text{minimize}} \ \sum_{k=1}^{n} f_k(\beta_k) + g_k(t_k) \quad (14)$$

$$\text{subject to } \frac{\beta_i \sum_{j=1}^{n} A_{ij} u_j + \bar{\delta}_i u_i}{(\bar{\Delta} + 1 - \varepsilon)\, u_i} \le 1, \quad (15)$$

$$(t_i + \bar{\delta}_i) / (\bar{\Delta} + 1) \le 1, \quad (16)$$

$$\bar{\Delta} + 1 - \overline{\delta}_i \le \bar{\delta}_i \le \bar{\Delta} + 1 - \underline{\delta}_i, \quad (17)$$

$$\underline{\beta}_i \le \beta_i \le \overline{\beta}_i, \ i = 1, \ldots, n. \quad (18)$$

Illustrations on Global AirTraffic

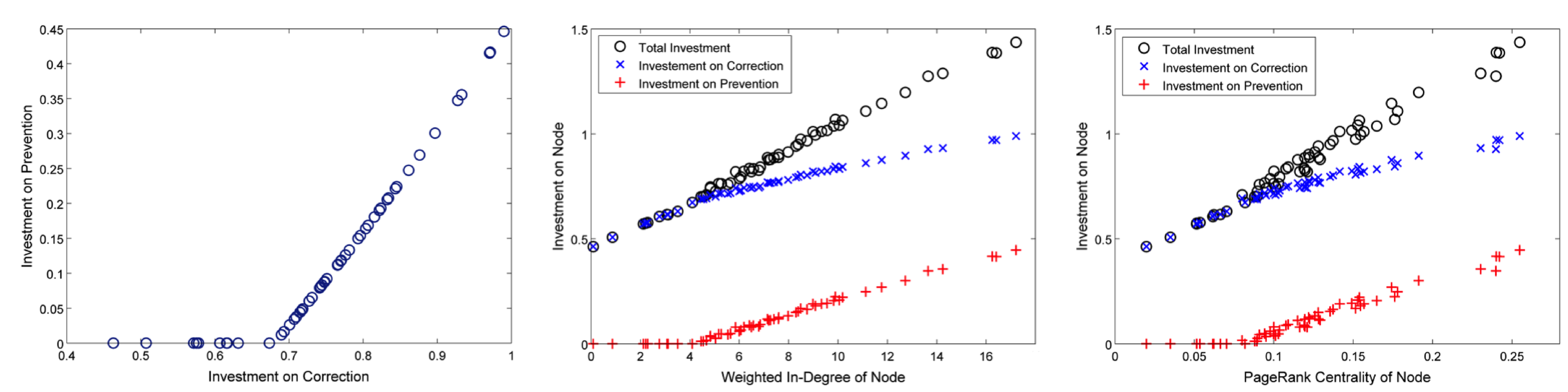

Figure 3: Results from the Budget-constrained resource allocation problem

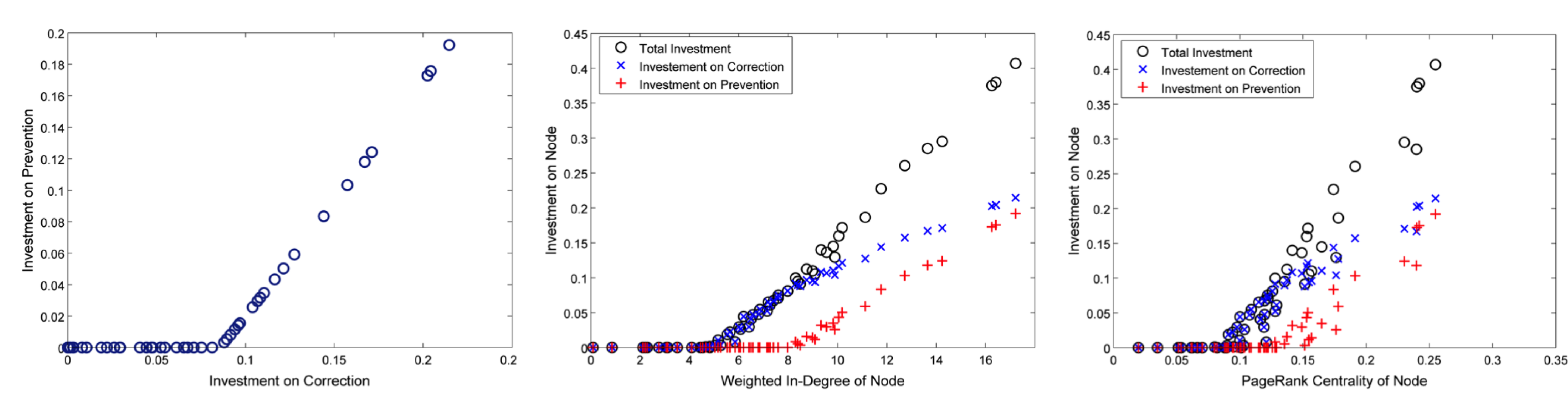

Figure 4: Results from the Rate-constrained resource allocation problem

Simulation parameters

- We applied results to a global air transportation network (56 airports with incoming passenger traffic $\ge$ 10 million per year)
- Bounds on infection and recovery rates used were chosen so that in the absence of protection resources, $\lambda_1(\beta_i A_{\mathcal{G}} - \delta_i I) = 0.1 > 0$

Conclusions

1. A convex framework indicates network centrality can guide allocation scheme, though it's not sufficient
2. Objective function determines whether or not one or both resources are applied
3. Observed (5 mppy and 8 mppy) thresholds for resource allocation

Challenges in CPNs

- Interdependence of different network layers - cyber (virtual) and physical makes them even more vulnerable.
- CPN configurations are not fixed; hence, not necessarily a one-shot allocation problem

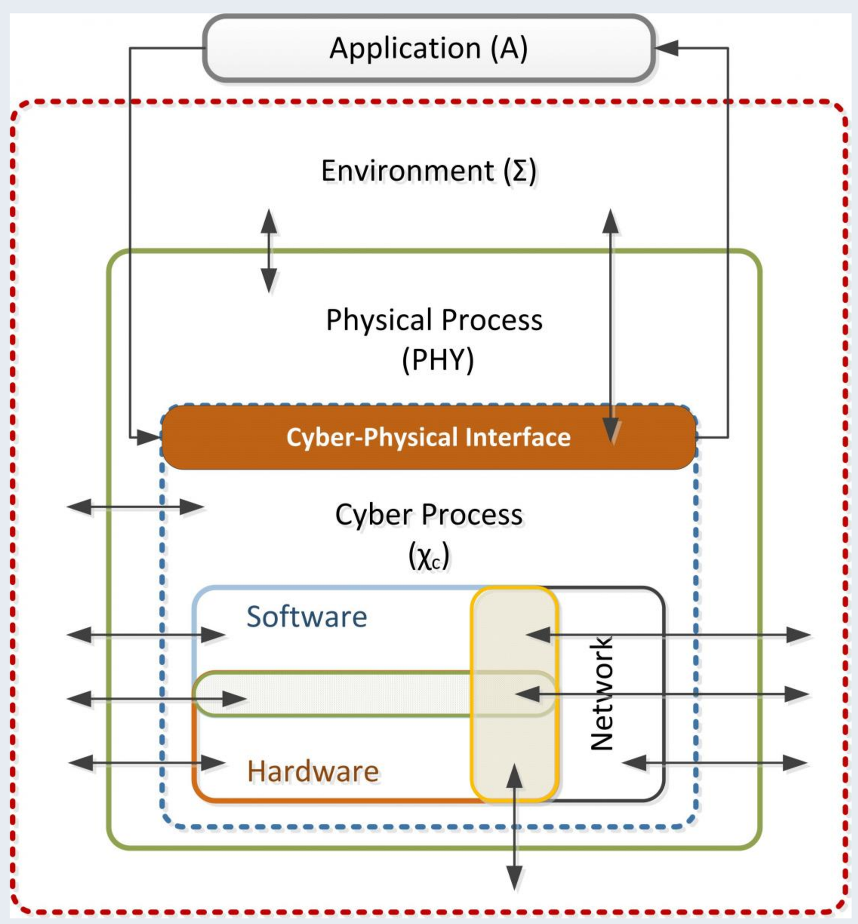

Figure 5: An illustration of layers in CPNs

- Efficient distributed solutions are needed
- Reliability and intermittent information in CPS.
- Allocation strategies that allow for graceful degradation?

References

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