Recent Results in Large Population Mean Field Stochastic Dynamic Control Theory: Consensus Dynamics Derived from the NCE Equations

Peter E. Caines

Department of Electrical & Computer Engineering
McGill University

Work with Minyi Huang, Roland Malhamé and Mojtaba Nourian

AFOSR Workshop

Washington, DC

USA

March 2009

Overview

- This research investigates:
  - Decision-making in stochastic dynamical systems with many competing agents
- Outline of contributions:
  - Nash Certainty Equivalence (NCE) Methodology
  - NCE for Linear-Quadratic-Gaussian (LQG) systems
  - Connection with physics of interacting particle (IP) systems
  - McK-V-HJB theory for fully nonlinear stochastic differential games
  - Invariance principle for controlled population behaviour
  - Models with interaction locality
  - Derivation the standard consensus dynamics from the NCE equations.

Some Facts and Implications

- Physics–Behavior of huge number of essentially identical infinitesimal interacting particles is basic to the formulation of statistical mechanics as founded by Boltzmann, Maxwell and Gibbs

- Game Theoretic Control System – Many competing agents
  - An ensemble of essentially identical players seeking individual interest
  - Individual mass interaction
  - Fundamental issue: how to relate individual actions to mass behavior?

Part I – Individual Dynamics and Costs

Individual dynamics:

$$dz_i = (a_i z_i + b u_i)dt + \alpha z^{(n)} dt + \sigma_i dw_i, \quad 1 \le i \le n. \tag{1}$$

- z_i: state of the ith agent
- $z^{(n)}$: the population mean $z^{(n)} \triangleq \frac{1}{n}\sum_{i=1}^{n} z_i$
- u_i: control
- w_i: noise (a standard Wiener process)
- n: population size

For simplicity: Take the same control gain b for all agents.

Part I – Individ. Dynamics and Costs (ctn)

Individual costs:

$$J_i(u_i, \nu_i) = E \int_0^\infty e^{-\rho t}[(z_i - \nu_i)^2 + r u_i^2] dt \qquad (2$$

We are interested in the case $\nu_i = \Phi(z^{(n)}) \triangleq \Phi(\frac{1}{n}\sum_{k=1}^{n} z_k)$

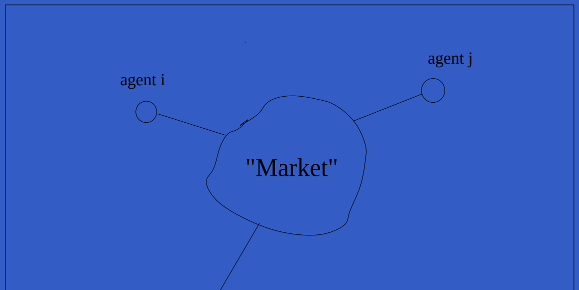
Φ: nonlinear and Lipschitz

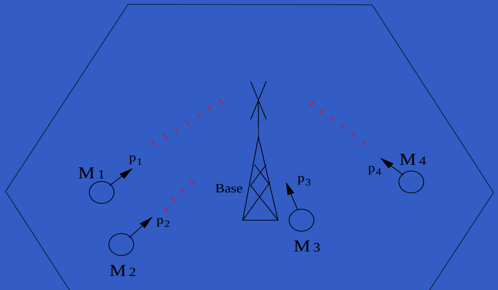
Main feature and Objective:

- Weak coupling via costs and dynamics
- Connection with IP Systems (for model reduction in McKean-Vlasov setting) with be clear later on
- Develop decentralized optimization

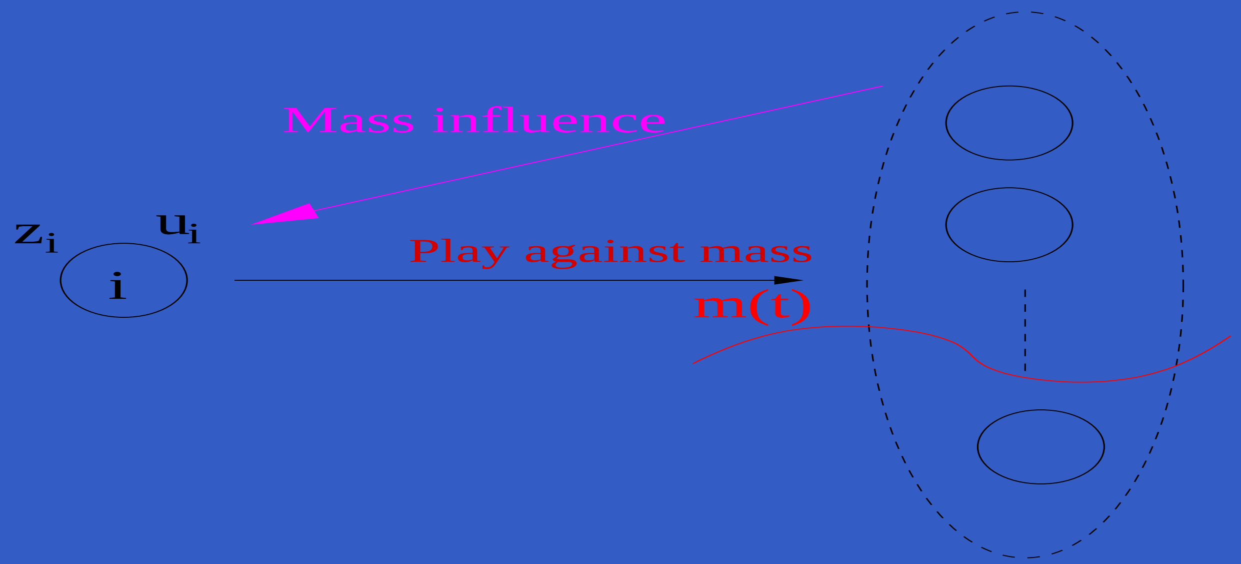
Part I – Motivational Background and Related Works

- Economic models (e.g., production output planning) where each agent receives average effect of others via Market (Lambson)
- Advertising competition game models (Erikson)
- Wireless network resource allocation (e.g., power control, HCM)
- Stochastic swarming (Morale et. al.); “selfish herd" (such as fish) reducing indiv. predation risk by joining group (Reluga & Viscido)
- Public health – Voluntary vaccination games (Bauch & Earn)
- Industry dynamics with many firms (Weintraub, Benkard, & Roy)
- Mathematical physics and finance (Lasry and Lions)
- Admission control in communication networks.





Part I – Control Synthesis via NCE



- Under large population conditions, the mass effect concentrates into a deterministic quantity $m(t)$.
- A given agent only reacts to the mass effect $m(t)$ and any other individual agent becomes invisible.
- Key issue is the specification of $m(t)$ and associated individual action - Look for certain consistency relationships

Part III – LQG-NCE Equation Scheme

Assume zero initial mean, i.e., $Ez_i(0) = 0$, $i \geq 1$. Based on population limit, the Fundamental NCE equation system:

$$\rho s_a = \frac{ds_a}{dt} + as_a - \frac{b^2}{r}\Pi_a s_a + \alpha \Pi_a \bar{z} - z^*, \quad (5$$

$$\frac{d\overline{z}_a}{dt} = (a - \frac{b^2}{r}\Pi_a)\overline{z}_a - \frac{b^2}{r}s_a + \alpha \bar{z}, \quad (6$$

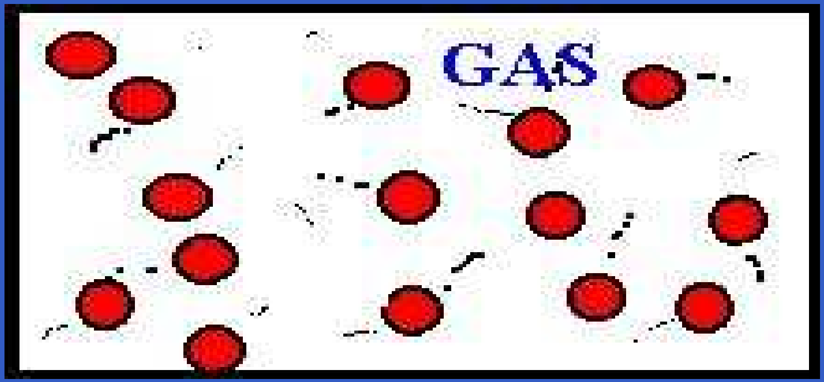
$$\overline{z} = \int_{\mathcal{A}} \overline{z}_a dF(a), \quad (7$$

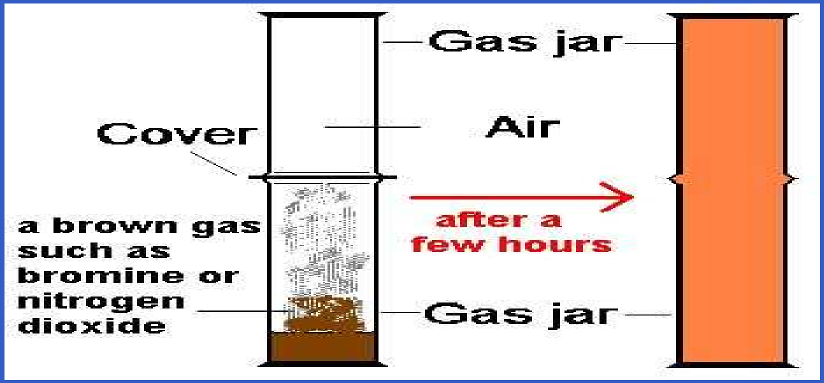
$$z^* = \Phi(\overline{z}). \quad (8$$

Basic idea behind NCE(z^*) with parameters $F(\cdot)$, a, b, α, r:

- Solve z^* tracking problem for one agent.
- Use popul. average $\overline{z}$ to approximate coupling term $\frac{1}{n}\sum_k^n z_k$.
- Individual action u_i is optimal response to z^*.
- Collectively produce same z^* assumed in first place.

Part V – Connection with Statistical Mechanics

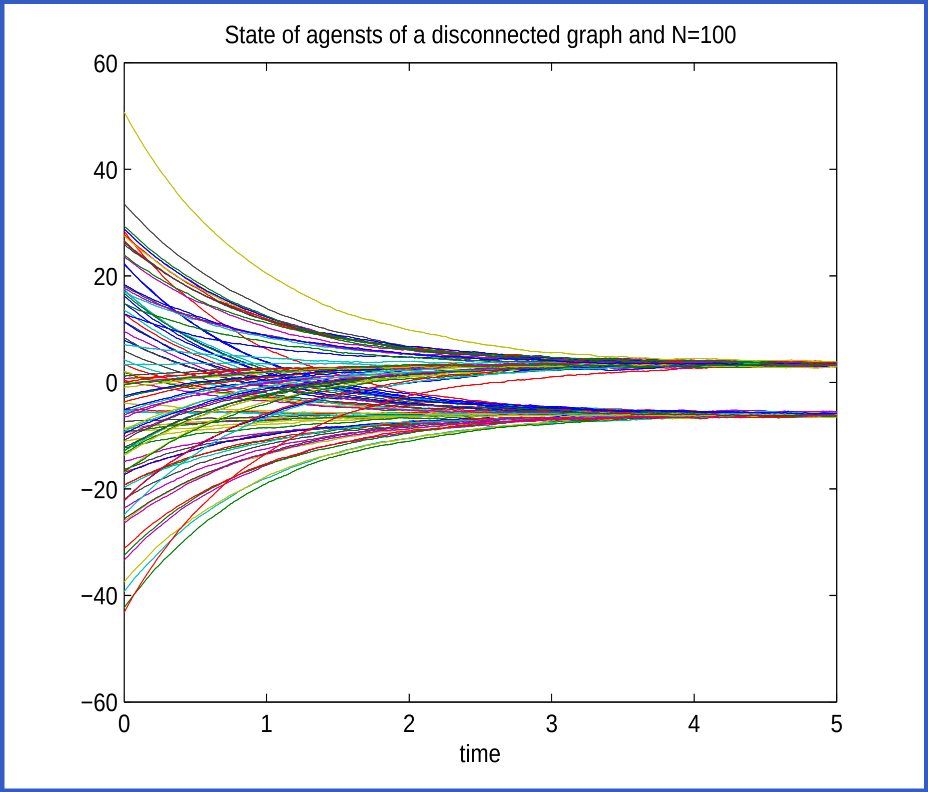




- Boltzmann PDE describing evolution of spatial-velocity ($x - v$) distribution $u(t, x, v)$ of huge number of gas particles
- Solution to spatially homogeneous Boltzman PDE (for $u(t, v)$) has a probabilistic interpret. via McKean's Markov system:
  - Generator depends on “current density" of the process
  - Thus, there exists a driving effect from the mass
    - This feature also appears in our diffusion based models, where current density affects the drift

Part IX – Simulations (cnt)

- For disconnected graphs we have the convergence of each group.



(c) Disconnected graph with two connected groups.

Concluding Remarks

- A theory for decentralized decision-making with many competing agents
- Control synthesis via NCE methodology. Consequences for Rational Expectations and Macroeconomic Policy?
- Existence of asymptotic equilibria (first in population then in time)
- Application to network call admission control (e.g. Ma, Malhamé, PEC)
- Ideas closely related to the physics of interacting particle systems.
- Suggest a convergence of control theory, multi-agent systems theory and statistical physics into a

  cybernetic-math physics synthesis

  for mass competitive-cooperative decision problems.