Verifying Continuous-time Stochastic Hybrid Systems via Mori-Zwanzig Model Reduction

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Introduction

We developed a method for verifying Continuous-time Stochastic Hybrid Systems (CTSHSs) using the Mori-Zwanzig model reduction method, whose behaviors are specified by Metric Interval Temporal Logic (MITL) formulas. By partitioning the state space of the CTSHS and computing the optimal transition rates between partitions, we provide a procedure to both reduce a CTSHS to a Continuous-Time Markov Chain (CTMC), and the associated MITL formulas defined on the CTSHS to MITL specifications on the CTMC. We prove that an MITL formula on the CTSHS is true (or false) if the corresponding MITL formula on the CTMC is robustly true (or false) under certain perturbations. In addition, we propose a stochastic algorithm to complete the verification.

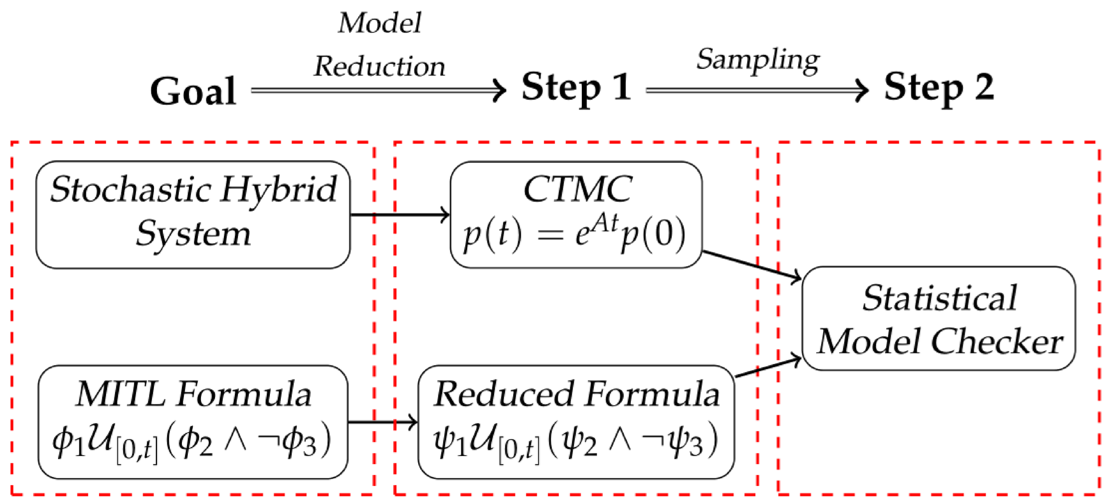

Figure 1. A roadmap for the work

System Formulation

The configuration space $Q \times \Omega$ of a Continuous-time Stochastic Hybrid System (CTSHS) is the Cartesian product of a set of discrete locations $Q = \{q_1, \dots, q_n\}$ and a continuous state space $\Omega = \mathbb{R}^d$. In a location $q \in Q$, the continuous state $x \in \Omega$ evolves by a stochastic differential equation

$$dx(t) = f(q_i, x)dt + g(q_i, x)dw_t,$$

where w_t is the standard white noise. It reduces to an ordinary differential equation when $g(q_i, x) = 0$.

Meanwhile, the system may switch to another location q_j and reset the continuous state to z by

$$(q_j, z) = h_j(q_i, x),$$

and the transition rate is given by $r_j(q_i, x)$.

Let $\tau(t) = (q(t), x(t))$ be a trajectory of the system, which obeys the time-evolving probability distribution $F(t, q, x)$. Then $F(t, q, x)$ satisfies the Fokker-Planck equation

$$\frac{\partial F(t, q_i, x)}{\partial t} = L\big(F(t, q_i, x)\big) = -\sum_{a=1}^{d} \frac{\partial}{\partial x_a}(f_a(q_i, x)F(t, q_i, x))$$
$$+\frac{1}{2}\sum_{a=1}^{d}\sum_{b=1}^{d} \frac{\partial^2}{\partial x_a \partial x_b}(g_a(q_i, x)g_b(q_i, x)F(t, q_i, x))$$
$$-\sum_{j=1}^{n} r_j(q_i, x)F(t, q_i, x)) + \sum_{\substack{h_i(q_j, y)\\=(q_i, x)}} r_i\big(q_j, y\big)F(t, q_j, y))$$

where L is the Fokker-Planck operator.

Similar to the Fokker-Planck equation for jump-diffusion process, the four terms on the right hand side stands for "drift", "diffusion", "jump-out" and "jump-in", respectively.

Metric Interval Temporal Logic

Generally, MITL is a decidable continuous-time extension of Linear Temporal Logic. To use MITL to describe the behavior of the trajectories of the CTSHS, we define an observable y on the system by

$$[y(\tau)](t) = \sum_{q \in Q} \int_{\Omega} \gamma(q, x)F(t, q, x)dx,$$

where τ is a trajectory of the system and $\gamma(q, x)$ is a given weight function.

The syntax of MITL is given recursively by

$\psi ::= \top \mid \bot \mid y_i {\sim} c_i \mid \neg\psi \mid \psi \wedge \phi \mid \psi \vee \phi \mid \psi \mathrm{U}_{(a,b)}\phi \mid \psi \mathrm{R}_{(a,b)}\phi$

where y_i is an observable of the system, $c \in \mathbb{R}$, $0 \le a < b \le \infty$ and $\sim\, \in \{>, <, \le, \ge\}$.

The satisfaction relation between a trajectory τ and an MITL formula ϕ is defined inductively by

$$\begin{aligned}
&\tau \vDash \top \qquad\qquad \tau \nvDash \bot\\
&\tau \vDash y_i {\sim} c \Leftrightarrow [y(\tau)](0) {\sim} c\\
&\tau \vDash \neg\phi \Leftrightarrow \tau \nvDash \phi\\
&\tau \vDash \phi \wedge \psi \Leftrightarrow \tau \vDash \phi \text{ and } \tau \vDash \psi\\
&\tau \vDash \phi \vee \psi \Leftrightarrow \tau \vDash \phi \text{ or } \tau \vDash \psi\\
&\tau \vDash \phi \mathrm{U}_{(a,b)}\psi \Leftrightarrow \exists t \in (a, b), (\tau, t) \vDash \psi\\
&\qquad\qquad \text{and } \forall s < t, (\tau, s) \vDash \phi\\
&\tau \vDash \phi \mathrm{R}_{(a,b)}\psi \Leftrightarrow (\forall t, (\tau, t) \vDash \psi) \text{ or}\\
&\qquad\qquad (\exists t \in (a, b), (\tau, t) \vDash \phi\\
&\qquad\qquad \text{and } \forall s \le t, (\tau, s) \vDash \psi),
\end{aligned}$$

where (τ, t) is the suffix of τ starting from t.

Model Reduction

1. Reducing the Dynamics

Let $S = \{s_1, s_2, \dots, s_l\}$ be a partition of the continuous state space Ω, namely,

1. Each s_i is nonempty, open and simply-connected
2. $\mu(\Omega \backslash \cup_{i=1}^{l} s_i) = 0$
3. $s_i \cap s_j = \emptyset$ for any $i \neq j$

Treating each partition as a discrete state, we can derive a Continuous-time Markov Chain (CTMC). The probability measures on the CTSHS and the CTMC are correlated by the projection $P: m(\mathrm{Q} \times \Omega) \to m(\mathrm{Q} \times \mathrm{S})$

$$p_{ij} = PF(q_i, x) = \int_{s_j} F(q_i, x)dx,$$

and the injection $R: m(\mathrm{Q} \times \Omega) \to m(\mathrm{Q} \times \mathrm{S})$

$$F(q_i, x) = Rp = \sum_{j=1}^{l} p_{ij}\mathbf{U}_{s_j}(x),$$

where $\mathbf{U}_{s_j}(x)$ is the uniform probability distribution on s_j.

To achieve the best approximation of the CTSHS, the transition rate matrix of the CTMC is given by

$$A = PLR$$

Specifically, the transition rate from state ij to state ab is

$$A_{abij} = \begin{cases} \displaystyle\int_{\partial s_j \cap \partial s_b} f(t, q_i, x)dx, & \text{if } a = i\\ \displaystyle\frac{1}{\mu(s_j)}\int_{s_j} r_a(t, q_i, x)\mathbf{I}_{h_a(t, q_i, x) \in s_b}dx, & \text{else}\end{cases}$$

for $i, a = 1, \dots, n$ and $j, b = 1, \dots, l$, where $\mathbf{I}_{h_a(t, q_i, x) \in s_b} = 1$ if $h_a(t, q, x) \in s_b$, and 0 otherwise.

$$\begin{tikzcd}
F(0,q,x) \arrow[r, "e^{Lt}"'] \arrow[d, dashed, "P"'] & F(t,q,x)\\
p(0) \arrow[r, "e^{At}"'] & p(t) \arrow[u, dashed, "Q"']
\end{tikzcd}$$

Figure 2. Model reduction error.

2. Reducing the MITL Formulas

As shown in Figure 2, the model reduction error of observable y at time t is

$$\Delta_y(t) = \left|\sum_{q \in Q} \int_{\Omega} \gamma(q, x)(e^{Lt} - Re^{At}P)F(0, q, x)dx\right|.$$

When the system is α-contractive for some $\alpha > 0$, namely there is $\beta \ge 1$ such that the inequality holds

$$\left|\sum_{q \in Q} \int_{\Omega} \gamma e^{Lt}\delta(q, x)dx\right| \le \beta e^{-\alpha}\left|\sum_{q \in Q} \int_{\Omega} \gamma\delta(q, x)dx\right|$$

for any L_1 function satisfying $\sum_{q \in Q} \int_{\Omega} \delta(q, x)dx = 0$, the model reduction error is bounded by

$$\Delta_y(t) \le \mathrm{A} + \frac{\beta B}{\alpha},$$

where

$$A = \sum_{q \in Q} \int_{\Omega} \gamma(q, x)(I - RP)F(0, q, x)dx,$$

$$B = \sup_{t \ge 0} \sum_{q \in Q} \int_{\Omega} \gamma(q, x)(L - RPL)F(t, q, x)dx,$$

are the reduction error for P and L respectively. They converge to 0 as we refine the partition.

This implies that, to verify an MITL formula ϕ on the trajectory τ of the CTSHS, it suffices to verify the MITL formula ψ derived by replacing each $y(t) > c$ with $y(t) > c + \mathrm{A} + \frac{\beta B}{\alpha}$ and each $y(t) < c$ with $y(t) < c + \mathrm{A} + \frac{\beta B}{\alpha}$ on the trajectory τ' of the CTMC.

Algorithm

Given the reduced CTMC C, the initial observation y_0 and a reduced MITL formula ψ, the statistical verification algorithm $\mathcal{A}^{\delta_1, \delta_2, y^{\mathrm{inv}}}(C, y_0, \psi, \alpha, \beta)$, together with the validity analysis, is presented by the following pseudo codes. We assume a priori knowledge of a unique invariant distribution of the reduced Markov process.

Input:

1. δ_1, δ_2 indifference parameters,
2. C input CTMC
3. y^{inv} invariant observation,
4. y_0 initial observation,
5. ψ MITL formula,
6. α, β error bounds.

Output:

- Yes, No, Unknown

Ensures:

- $P[\mathrm{out} = \mathrm{Yes} \mid \tau' \nvDash \psi] \le \alpha$
- $P[\mathrm{out} = \mathrm{No} \mid \tau' \vDash \psi] \le \alpha$
- $P\left[\mathrm{out} = \mathrm{Unknown} \,\middle|\, \begin{matrix}\forall\tau \bullet \|\tau' - \tau\| \le \delta_1 \Rightarrow \tau \vDash \psi\\ \forall\tau \bullet \|\tau' - \tau\| \le \delta_1 \Rightarrow \tau \nvDash \psi\end{matrix}\right] \le \alpha + \beta$

Procedure:

1. Use $close\left(y(T), y^{\mathrm{inv}}, \frac{3\alpha}{4}, \delta_2\right)$ to find T such that $\|y(T) - y^{\mathrm{inv}}\| \le \delta_2$
2. Find Δ such that $\forall t \in [0, T], t' \in [t - \Delta, t + \Delta] \cap [0, T]$
   1. $y_i(t) - c > \frac{\delta_1}{3} \Rightarrow y_i(t') > 0$
   2. $y_i(t) - c < \frac{\delta_1}{3} \Rightarrow y_i(t') < 0$
   3. $|y_i(t) - c| < \frac{2\delta_1}{3} \Rightarrow |y_i(t') - c| < \delta_1$
3. Partition $[0, T]$ into disjoint intervals of length 2Δ
4. For each t middle of an interval, let
   1. $res_1 \leftarrow \mathcal{A}_1^{\delta_1/3}\left(y_i(t), c + \frac{\delta_1}{3}, \alpha', \beta'\right)$
   2. $res_2 \leftarrow \mathcal{A}_1^{\delta_1/3}\left(y_i(t), c - \frac{\delta_1}{3}, \alpha', \beta'\right)$
5. Use res_1, res_2, and step 2 to categorize the intervals
6. Construct timed automaton $T_{C,AP}$ using steps 1 & 5

   /* $Lang(T_{C,AP})$ contains the exact signal and more */

1. **if** $Lang(T_{C,AP}) \cap Lang(T_{\psi}) = \emptyset$ **then return** No
2. **if** $Lang(T_{C,AP}) \cap Lang(T_{\neg\psi}) = \emptyset$ **then return** Yes
3. **return** Unknown

Conclusions and Future Work

In this work, we proposed a framework of using metric interval temporal logic formulas to describe the behavior of Continuous-time Stochastic Hybrid Systems and a method of using the Mori-Zwanzig model reduction method to verify the temporal logic formulas. Specifically, We proved that the problem of verifying the temporal logic formulas on the CTSHS can be transformed to the problem of verifying a slightly stronger formulas on the CTMC and proposed a sampling-based method to finish the verification. We have implemented this method in a Billiard problem to verify the reachability property. In the future, we will implement this method to more real-world applications, such as powertrain systems.

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