On ellipsoidal techniques for reachability analysis Part II. Internal Approximations Box-valued constraints *

A.B.Kurzhanski
Moscow State University
P.Varaiya
University of California at Berkeley

Abstract

Following Part I, this paper continues to describe the calculation of the reach sets and tubes for linear control systems with time-varying coefficients and ellipsoidal hard bounds on the controls and initial states. It deals with parametrized families of internal ellipsoidal approximations constructed such that they touch the reach sets at *every point* of their boundary at any instant of time (both from outside and inside respectively). The surface of the reach tube would then be entirely covered by curves that belong to the approximating tubes. This allows exact parametric representation of reach tubes through families of internal ellipsoidal tubes as compared with earlier methods based on constructing one or several isolated approximating tubes. The method of external and internal ellipsoidal approximations is then propagated to systems with box-valued hard bounds on the controls and initial states. The approach opens new routes to the arrangement of efficient numerical algorithms.

Introduction

This papers continues description of ellipsoidal approximations to reach sets for continuous-time systems. Such problems arise in many applied problems of control and computation, being an object of special interest [19], [4], [7], [6], [14], [17].

It turned that ellipsoidal methods make possible *exact* representations of the reach sets and tubes for linear systems through parametrized families of external and internal ellipsoids if these are constructed following the techniques of [13]. But to ensure effective calculation, it is important to single out the families of "tight" ellipsoidal approximations to the reach tube that would touch its surface at *every point* along specially selected cyrves and would thus *totally cover* this tube. A crucial point is also to indicate such a parametrized variety of curves along which the respective calculation

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could be done *recurrently in time.* A positive answer to the latter problem is given in PartI of this paper for external ellipsoidal approximations which were investigated in detail.

This paper shows that similar properties are also true for *internal approximations* which are often required whenever one has to deal with *guaranteed performance.* This is a more difficult problem, though. A single volume-optimal internal ellipsoid was studied in [2]. Special types of internal ellipsoidal approximations were suggested in [1]. However, it was indicated in [13] that an exact representation of reach sets and reach tubes is indeed possible through the union of a family of internal ellipsoids. The important pending question was how to effectively compute a family of tight internal approximations of reach tubes or their neighborhoods through such ellipsoidal-valued functions that would touch their boundary of from inside at *any point* on its surface.

Thus, in the present Part II of this paper we study the following question: given a reach tube (or its neighborhood) and any smooth curve on its surface, does there exist an ellipsoidal-valued tube that satisfies the following two properties: on one hand, the ellipsoidal tube is contained *inside* the reach tube, being an internal approximation, and on the other it touches the boundary of the reach tube precisely along the prespecified continuous curve on its surface? Here it is shown that the solution to the problem exists for any given curve. However, the computational burden for the calculation of the solution may be heavy due to additional recalculations required to be done "afresh" at each instant of time. It is shown that similar to the "external" case there exists a family of "good" curves which allow the solution of the indicated problem recurrently, without such additional recalculation. This again is when the given curve is *a system trajectory.* By covering the entire surface of the reach tube with such curves, we are then also able to produce the whole reach tube, through the solutions of the internal ellipsoidal approximation problem while minimizing the computational burden.

In order to treat the "internal" case this paper introduces new relations for internal ellipsoids, which are different from those introduced in books [1], [2], [13].

Let us start with th whole issue of internal ellipsoids by introducing some relations different from those of either [1], [2], [13].

Throughout the present Part II of the paper we use the notations introduced in Part I.

1 The internal ellipsoids

Consider first the sum of two ellipsoids $\mathcal{E}(0, Q_1), \mathcal{E}(0, Q_2)$, taking as an estimate the ellipsoid $\mathcal{E}(0, Q_[S])$, where

$$Q_-[S] = (S_1 Q_1^{1/2} + S_2 Q_2^{1/2})'(S_1 Q_1^{1/2} + S_2 Q_2^{1/2}),$$

and S_1, S_2 are orthogonal matrices [5], so that $S_1' S_1 = I = S_2' S_2$.

Then

$$(\rho(l|\mathcal{E}(0, Q_-[S])))^2 = (l, Q_-[S]l)$$
$$= (l, Q_1 l) + (l, Q_2 l) + 2(S_1 Q_1^{1/2} l, S_2 Q_2^{1/2} l)$$
$$\leq (l, Q_1 l) + (l, Q_2 l) + 2(l, Q_1 l)^{1/2}(l, Q_2 l)^{1/2} = ((l, Q_1 l)^{1/2} + (l, Q_2 l)^{1/2})^2$$

for any vector $l \in \mathbb{R}^n$ with equality attained if and only if

$$S_1 Q_1^{1/2} l = k S_2 Q_2^{1/2} l$$

for some value k. The last assertion follows from the Hölder inequality.

Similarly, for the sum of m ellipsoids $\mathcal{E}(0, Q_i), i = 1, ..., m$ we arrive at the proposition.

Theorem 1.1 *The following formula is true:*

$$\mathcal{E}_-(0, Q[S(m)]) \subseteq \sum_{i=1}^{m} \mathcal{E}_i(0, Q_i),$$

where

$$Q[S(m)] = (\sum_{i=1}^{n} S_i Q^{1/2})'(\sum_{i=1}^{m} S_i Q^{1/2}),$$

$S[(m)] = \{S_1, \ldots, S_m\}$ *and* $S_i' S_i = I$ *are any orthogonal matrices.*

Indeed, using the Hölder inequality, we come to the relations:

$$(\rho(l|\mathcal{E}(0, Q_-[S(m)])))^2 = (l, Q_-[S(m)]l) = \sum_{i=1}^{m}(l, Q_i l) + \sum_{i \neq j}^{m}(S_i Q_i^{1/2} l, S_j Q_j^{1/2} l)$$

$$\leq \sum_{i=1}^{m}(l, Q_i l) + \sum_{i \neq j}(l, Q_i l)^{1/2}(l, Q_j l)^{1/2} = (\sum_{i=1}^{m}(l, Q_i l)^{1/2})^2$$

for any vector $l \in \mathbb{R}^n$. This proves the theorem.

Here equality is attained for a given vector l if and only if $S_i Q_i^{1/2} l = k_{ij} S_j Q_j^{1/2} l, \quad k_{ij} = k_{ji}^{-1}$, for some values k_{ij} (that depend on l). This happens, in its turn, iff there exists a vector $p \neq 0$ and and an array of numbers λ_i such that

$$S_i Q_i^{1/2} l = \lambda_i p, \quad i = 1, \ldots, m. \tag{1}$$

Therefore, one may formulate the next proposition.

Theorem 1.2 *The inequality*

$$(l, Q[S(m)]l) = (l, (\sum_{i=1}^{m} S_i Q^{1/2})'(\sum_{i=1}^{m} S_i Q^{1/2})l) \leq \sum_{i=1}^{m}(l, Q_i l)^{1/2}, \tag{2}$$

true for any $l \in \mathbb{R}^n$ *and any array* $S[m]$ *of orthogonal matrices* $S_i, i = 1, \ldots, m$, *turns into an equality for a given* $l \in \mathbb{R}^n$ *if and only if the matrices* S_i *are such that (1) is fulfilled for the given* l *for some vector* p *and numbers* λ_i, $i = 1, \ldots, m$.

The last theorem indicates *the tightness conditions* for the internal ellipsoidal approximation $\mathcal{E}_-(0, Q[S(m)])$ of the sum of ellipsoids $\mathcal{E}(0, Q_i)$.

Remark 1.1 The previous relations were derived for the sum of ellipsoids centered at point $q_i = 0$. The result remains true, however, if we deal with ellipsoids $\mathcal{E}(q_i, Q_i)$, where it may be that $q_i \neq 0$. Then one should just substitute $\mathcal{E}_-(0, Q_-[S(m)])$ by $\mathcal{E}_-(q(m), Q_-[S(m)])$, where

$$q(m) = \sum_{i=1}^{m} q_i. \tag{3}$$

We now look at the internal approximation of the sum

$$\mathcal{E}(0, Q_0) + \int_{t_0}^{t} \mathcal{E}(0, Q(s))ds,$$

of an ellipsoid $\mathcal{E}(0, Q_0)$ and a set-valued integral of an ellipsoidal-valued function $\mathcal{E}(0, Q(s))$ with matrix function $Q(s) > 0$ continuous in t. Taking the Riemanian sum

$$I(\Sigma_N) = \sum_{i=1}^{n} \mathcal{E}(q(\tau_i), Q(\tau_i))\sigma_i, \tag{4}$$

generated by partition

$$\Sigma_N = \{\tau_0 = t_0, \tau_1, \ldots, \tau_N = t\}, \quad \sigma_i = \tau_i - \tau_{i-1}, \;\; i = 1, \ldots, N,$$

we may apply the results of Theorems 1.1, 1.2. We start by constructing the matrix

$$Q_-^{(N)}(t) = \left(S_0 Q_0^{1/2} + \sum_{i=1}^{N} S_i Q_i^{1/2}(\tau_i)\sigma_i \right)' \left(S_0 Q_0^{1/2} + \sum_{i=1}^{N} S_i Q_i^{1/2}(\tau_i)\sigma_i \right), \tag{5}$$

which, as we may observe, due to these theorems, satisfies the inequality

$$(l, Q_-^{(N)}(t)l) \leq ((l, Q_0 l)^{1/2} + \sum_{i=1}^{N}(l, Q(\tau_i)l)^{1/2}\sigma_i)^2, \tag{6}$$

whatever be vector $l \in \mathbb{R}^n$. An equality may be reached here for a given $l \in \mathbb{R}^n$ iff the orthogonal matrices S_0, S_i are selected such that they satisfy the relations

$$S_i(\tau_i)Q^{1/2}(\tau_i)l = \lambda(\tau_i)S_0 Q_0^{1/2} l \tag{7}$$

for some values $\lambda(\tau_i), i = 1, \ldots, N$.

One may figure out immediately from (7) that these values actually are

$$\lambda_i(\tau) = (l, Q(\tau_i)l)^{1/2}(l, Q_0 l)^{-(1/2)}, \tag{8}$$

Further on, passing to the limit in (4)-(6), with

$$N \to \infty, \;\; \sigma[N] = \max\{\sigma_i | i = 1, \ldots, N\} \to 0,$$

we come to the following conclusion.

Theorem 1.3 *(i) The following inclusion is true*

$$\mathcal{E}(0, Q_-(t)) \subseteq \mathcal{E}(0, Q_0) + \int_{t_0}^{t} \mathcal{E}(0, Q(\tau))d\tau,$$

whatever be the matrix

$$Q_-(t) = \left(S_0 Q_0^{1/2} + \int_{t_0}^{t} S(\tau)Q(\tau)^{1/2}d\tau \right)' \left(S_0 Q_0^{1/2} + \int_{t_0}^{t} S(\tau)Q(\tau)^{1/2}d\tau \right), \tag{9}$$

where $S_0 S_0' = I, S'(\tau)S(\tau) \equiv I$ *are orthogonal matrices that vary continuously in time.*

(ii) For a given vector $l \in \mathbb{R}^n$ relation (7) turns into an equality iff matrices $S_0, S(\tau)$ may be chosen such that equality

$$S(\tau)Q^{1/2}(\tau)l = \lambda(\tau)S_0Q_0^{1/2}l \tag{10}$$

is fulfilled for all $\tau \in [t_0, t]$ for some scalar function $\lambda(\tau)$.

One may note immediately that function $\lambda(\tau)$ of (10) is actually

$$\lambda(\tau) = (l, Q(\tau)l)^{1/2}(l, Q_0 l)^{-(1/2)}.$$

The proof of this theorem follows by direct limit transition in (4) in view of the inequality

$$\left(l, \left(S_0Q_0^{1/2} + \int_{t_0}^{t} S(\tau)Q^{1/2}(\tau)d\tau\right)' \left(S_0Q_0^{1/2} + \int_{t_0}^{t} S(\tau)Q^{1/2}(\tau)d\tau\right) l\right) \tag{11}$$

$$\leq \left(\int_{t_0}^{t} (l, Q(\tau)l)^{1/2}d\tau + (l, Q_0 l)^{1/2}\right)^2$$

that follows from a limit transition in (5),(6), with $\sigma[N] \to 0, N \to \infty$.

Remark 1.2. For ellipsoids $\mathcal{E}(q_0, Q_0), \mathcal{E}(q(t), Q(t))$ with nonzero centers $q_0, q(t)$ the results of Theorem 7.3 remain true with substitution of $\mathcal{E}_-(0, Q_-(t))$ by $\mathcal{E}(q_-(t), Q_-(t))$, where

$$q_-(t) = q_0 + \int_{t_0}^{t} q(s)ds. \tag{12}$$

The functions $q_-(t), Q_-(t)$ may be described by differential equations. Thus, differentiating $Q_-(t)$, we have

$$\dot{Q}_-(t) = (S(t)Q^{1/2}(t))'Q_*(t) + Q_*(t)(S(t)Q^{1/2}(t)),$$

where

$$Q_*(t) = S_0Q_0^{1/2} + \int_{t_0}^{t} S(\tau)Q^{1/2}(\tau)d\tau$$

Introducing notation

$$H(t) = Q_*^{-1}(t)S(t)Q^{1/2}(t) = Q_*^{-1}(t)\dot{Q}_*(t),$$

we further come to equation

$$\dot{Q}_-(t) = H'(t)Q_-(t) + Q_-(t)H(t), \quad Q(t_0) = Q_0. \tag{13}$$

A differentiation of (12) also gives

$$\dot{q}_-(t) = q(t), \quad q_-(t_0) = q_0. \tag{14}$$

This may be summarized in the assertion.

Lemma 1.1 *The ellipsoid $\mathcal{E}(q_-(t), Q_-(t))$ may be described by differential equations (13),(14).*

Relation (13) allows an expansion

$$Q_-(t+\sigma) = Q_-(t) + \sigma(\dot{Q}'_* Q_*(t) + Q_*(t)'\dot{Q}_*), + o(\sigma) \tag{15}$$

where $o(\sigma)/\sigma \to 0$ with $\sigma \to 0$, or an equivalent relation

$$(l, Q_-(t+\sigma)l) = (l, Q_-(t)l) + \sigma^2(l, (\dot{Q}'_* Q_*(t) + Q_*(t)'\dot{Q}_*)l) + o(\sigma, l), \tag{16}$$

where $o(\sigma, l)$ stands for a function that satisfies $o(\sigma, l)/\sigma \to 0$ with $\sigma \to 0$, uniformly in $\{l : (l, l) \leq 1\}$. On the other hand, the relation for $Q_-(t)$ of Theorem 1.3 gives

$$\rho^2(l|\mathcal{E}(0, Q_-(t+\sigma))) = (l, Q_-(t+\sigma)l) = (l, Q_-(t)l) + \tag{17}$$
$$+\left(l, \left(\int_t^{t+\sigma} S(\tau)Q(\tau)^{1/2}d\tau\right)' Q_*(t) + Q'_*(t)\left(\int_t^{t+\sigma} S(\tau)Q(\tau)^{1/2}d\tau\right)l\right) + o_1(t, l)$$
$$\subseteq (\rho(l|\mathcal{E}(0, Q_0)) + + \int_{t_0}^{t+\sigma} \rho(l|\mathcal{E}(0, Q(\tau)))d\tau)^2 = \rho^2(l|\mathcal{X}[t]),$$

with equality attained for a given vector l (relative to terms of order > 1 in σ) iff there exists a number $\lambda(t)$, such that

$$\left(\int_t^{t+\sigma} S(\tau)Q(\tau)^{1/2}d\tau\right)l = \lambda(t)Q_*(t)l.$$

As one may observe, here

$$\lambda^2(t) = (l, Q(t)l)/(l, Q_-(t)l) + o_2(t, l).$$

and also

$$\int_t^{t+\sigma} S(\tau)Q(\tau)^{1/2}d\tau = \dot{Q}_*(t) + o_3(t, l).$$

Since equalities (16),(17) ensure, (relative to terms of order > 1 in σ) that $\mathcal{E}(0, Q_-(t+\sigma))$ touches $\mathcal{X}[t+\sigma]$ at point of support of vector l, we may conclude that $\dot{Q}_*(t)l = \lambda(t)Q_*l$ or

$$H(t)l = Q_*^{-1}\dot{Q}_*(t)l = \lambda(t)l, \tag{18}$$

where $\lambda(t) = (l, Q(t)l)^{1/2}/(l, Q_-(t)l)^{1/2}$.

Now we may proceed with the calculation of tight internal approximations.

2 Reachability sets. Internal approximations.

We start with the definition of internally tight ellipsoids.

Definition 2.1 *An internal approximation* $\mathcal{E}_-$ *is* **tight** *in the class* $\mathbf{E}_-$*, if for any ellipsoid* $\mathcal{E} \in \mathbf{E}_-$*,* $\mathcal{X}[t] \supseteq \mathcal{E} \supseteq \mathcal{E}_-$ *implies* $\mathcal{E} = \mathcal{E}_-$*.*

This paper is concerned with internal approximations, where class $\mathbf{E} = \{\mathcal{E}_-\}$ *is described within the following definition* (assuming $B = I$ as mentioned above).

Definition 2.2 *The class* $\mathbf{E}_- = \{\mathcal{E}_-\}$ *consists of ellipsoids that are of the form* $\mathcal{E}_-[t] = \mathcal{E}(x^\star, Q_-[t])$, *where* $x^\star(t)$ *satisfies the equation*

$$\dot{x}^\star = A(t)x^\star + q(t),\ x^\star(t_0) = x^0,\ t \geq t_0,$$

and $Q_-(t)$ *is of the form (9).*

Here $Q^0, Q(\tau), \tau \in [t_0, t]$ *are any positive definite matrices with function* $Q(\tau)$ *continuous,* $S(\tau)$ *are any orthogonal matrices with* $S(\tau)$ *continuous,* $q(t)$ *is any continuous function.*

In particular, this means that if ellipsoid $\mathcal{E}(p^0, P^0) \subseteq \mathcal{X}[t]$ is tight in $\mathbf{E}_-$, then there exists no other ellipsoid of type $\mathcal{E}(p^0, kP^0), k > 1$ that satifies the inclusions $\mathcal{X}[t] \supseteq \mathcal{E}(p^0, kP^0) \supseteq \mathcal{E}(p^0, P^0)$ (ellipsoid $\mathcal{E}(p^0, P^0)$ *touches* set $\mathcal{X}[t]$).

Definition 2.3 *We shall further say that internal ellipsoids are* **tight** *if they are tight in* $\mathbf{E}_-$.

We actually further deal only with ellipsoids $\mathcal{E}_- \in \mathbf{E}_-$.

The class $\mathbf{E}_-$ is rich enough to arrange effective approximation schemes, though it does not include all the possible ellipsoids.

A justification for using class $\mathbf{E}_-$ is due to the propositions of Theorem 1.3 which also gives conditions for the internal ellipsoids $\mathcal{E}(0, Q_-(\tau))$ to be *tight* in the previous sense.

Let us now return to equation

$$\dot{x} = A(t)x + B(t)u,\quad t_0 \leq t \leq t_1, \tag{19}$$

at first with $B(t) \equiv I$. (See Remark 2.2 at the end of this section). Then the problem consists in finding the internal ellipsoid $\mathcal{E}(q_-(t), Q_-(t))$ for the set

$$\mathcal{X}[t] = G(t, t_0)\mathcal{E}(q_0, Q_0) + \int_{t_0}^{t} G(t, \tau)\mathcal{E}(q(t), Q(\tau))d\tau.$$

The formula of Theorem 7.3 for the matrix $Q_-(t)$ will now have the form [1]

$$\begin{aligned} Q_-(t) &= \\ &= G(t, t_0)\left(Q_0 S_0'(t_0) + \int_{t_0}^{t} G(t_0, \tau)Q^{1/2}(\tau)S'(\tau)d\tau\right) \times \\ &\left(S_0(t_0)Q_0 + \int_{t_0}^{t} S(\tau)Q^{1/2}(\tau)G'(t_0, \tau)\right)G'(t, t_0), \end{aligned} \tag{20}$$

and

$$q_-(t) = G(t, t_0)q_0 + \int_{t_0}^{t} G(t, \tau)q(\tau)d\tau. \tag{21}$$

Theorem 1.3 now transforms into the following.

[1] Since for a matrix-valued map X we have $X\mathcal{E}(q, Q) = \mathcal{E}(Xq, XQX')$.

Theorem 2.1 *The internal ellipsoids for the reach set $\mathcal{X}[t]$ satisfy relation*

$$\mathcal{E}(q_-(t), Q_-(t)) \subseteq \tag{22}$$

$$\mathcal{E}(G(t,t_0)q_0, G(t,t_0)Q_0G'(t,t_0)) + \mathcal{E}\left(\int_{t_0}^t G(t,\tau)q(\tau)d\tau, \int_{t_0}^t G(t,\tau)Q(\tau)G'(t,\tau)d\tau\right) = \mathcal{X}[t],$$

where $Q_-(t_0), q_-(t_0)$ are given by (20),(21), with $S_0, S(\tau)$ being any orthogonal matrices and $S(\tau)$ continuous in time.

The tightness conditions now transfer into the next proposition.

Theorem 2.2 *For a given instant t the internal ellipsoid $\mathcal{E}(q_-(t), Q_-(t))$ will be tight and will touch $\mathcal{X}[t]$ at the point of support x^* of the tangent hyperplane generated by given vector l^*, namely,*

$$\rho(l^*|\mathcal{X}[t]) = (l^*, q_-(t)) + \int_{t_0}^t (l^*, G(t,\tau)Q(\tau)G'(t,\tau)l^*)^{1/2}d\tau = \tag{23}$$

$$\rho(l^*|\mathcal{E}(q_-(t), Q_-(t))) = (l^*, q_-(t)) + (l^*, Q_-(t)l^*)^{1/2} = (l^*, x^*),$$

iff $S_0, S(\tau)$ satisfy the relation

$$S(\tau)Q^{1/2}(\tau)G'(t,\tau)l^* = \lambda(\tau)S_0Q_0^{1/2}G'(t,t_0)l^*, \ \ t_0 \leq \tau \leq t, \tag{24}$$

for some function $\lambda(\tau)$.

Direct calculation indicates the following.

Lemma 2.1 *The function $\lambda(\tau)$ of Theorem 2.2 is given by*

$$\lambda(\tau) = (l^*, G(t,\tau)Q(\tau)G'(t,\tau)l^*)^{1/2}(l^*, G(t,t_0)Q_0G'(t,t_0)l^*)^{(-1/2)}, \ t_0 \leq \tau \leq t. \tag{25}$$

The previous Theorems 2.1, 2.2 were formulated *for a fixed instant of time t and a fixed support vector l^**. It is important to realize what would happen if l^* varies in time.

Problem 2.1. *Given a vector function $l^*(t)$, continuously differentiable in t, find an internal ellipsoid $\mathcal{E}(q_-(t), Q_-(t)) \subseteq \mathcal{X}[t]$ that would ensure* **for all** *$t \geq t_0$, the equality*

$$\rho(l^*(t)|\mathcal{X}[t]) = \rho(l^*(t)|\mathcal{E}(q_-(t), Q_-(t)) = (l^*(t), x^*(t)), \tag{26}$$

so that the supporting hyperplane for $\mathcal{X}[t]$ generated by $l^*(t)$, namely, the plane $(x - x^*(t), l^*(t)) = 0$ that touches $\mathcal{X}[t]$ at point $x^*(t)$, would also be a supporting hyperplane for $\mathcal{E}(q_-(t), Q_-(t)$ and touch it at the same point.

In order to solve this problem, we shall refer to Theorems 1.1, 1.2. However, the functions $S(\tau), \lambda(\tau)$ used for the parametrization in (20),(24), should now be functions of two variables, namely, of τ, t, since the requirement is that relation (26) should now hold for all $t \geq t_0$ (and therefore S_0 should also depend on t). We may therefore still apply Theorems 1.1, 1.2 but now with $S_0, S(\tau), \lambda(\tau)$ substituted by $S_{0t}, S_t(\tau), \lambda_t(\tau)$.

Theorem 2.3 *With $l = l^*(t)$ given, the solution to Problem 2.1 is an ellipsoid $\mathcal{E}(q_-(t), Q_-(t))$, where*

$$Q_-(t) = \tag{27}$$

$$= G(t,t_0)\Big(Q_0^{1/2}S'_{0t}(t_0) + \int_{t_0}^{t} G(t_0,\tau)Q^{1/2}(\tau)S'_t(\tau)d\tau\Big)\times$$

$$\Big(S_{0t}(t_0)Q_0^{1/2} + \int_{t_0}^{t} S_t(\tau)Q^{1/2}(\tau)G'(t_0,\tau)\Big)G'(t,t_0).$$

with $S_0, S_t(\tau)$ satisfying relations

$$S_t(\tau)Q^{1/2}(\tau)G'(t,\tau)l^*(t) = \lambda_t(\tau)S_{0t}Q_0^{1/2}G'(t,t_0)l^*(t), \tag{28}$$

and $S'_{0t}S_{0t} = I;\ S'_t(\tau)S_t(\tau) \equiv I$ for all $t \geq t_0,\ \tau \in [t_0,t]$, where

$$\lambda_t(\tau) = (l^*(t), G(t,\tau)Q(\tau)G'(t,\tau)l^*(t))^{1/2}(l^*(t), G(t,t_0)Q_0G'(t,t_0)l^*(t))^{(-1/2)}. \tag{29}$$

The proof follows by direct substitution. The latter relations are given in a "static" form and Theorem 1.3 indicates that the calculation of parameters $S_{0t}, S_t(\tau), \lambda_t(\tau)$ has to be done "afresh" for every new instant of time t. We shall now investigate whether the calculations can be made in a recurrent form, without having to perform the additional recalculation.

Remark 2.1. In all the ellipsoidal approximations considered in this paper the center of the aprroximating ellipsoid is always the same, being given by $q_-(t)$ of (21). The discussions shall therefore actually concern only the relations for $Q_-(t)$.

Remark 2.2. In the case of $B(t) \neq I$ we formally just have to substitute $Q(t)$ for $B(t)Q(t)B'(t)$, $Q^{1/2}(t)$ for $B(t)Q^{1/2}(t)$ and $q(t)$ for $B(t)q(t)$ in all the relations of this and later sections, starting the calculation process from time $t \geq t_0 + \sigma,\ \sigma > 0$ rather then from t_0. The controllability assumption of section 1 ensures that $int\mathcal{X}[t] \neq \emptyset$ and that the ellipsoids $\mathcal{E}(q_(t), Q_-(t))$ are nondegenerate.

Remark 2.3. The results of the last two sections are also true for degenerate ellipsoids $\mathcal{E}(q^0, Q^0),\ \ \mathcal{E}(q(t), Q(t))$. This will further allow to treat systems with box-valued constraints.

3 Reachability Tubes. Internal Approximations.

Let us start with a particular function $l^*(t)$, namely, the one that satisfies [2]

Assumption 3.1 *The function $l^*(t)$ is of the following form $l^*(t) = G(t_0,t)l$, with $l \in \mathbb{R}^n$ given. For the time-invariant case $l^*(t) = (\exp(-A'(t)(t-t_0))l$.*

Substituting $l^*(t)$ in (28),(29), we observe that the relations for calculating $S_t(\tau), \lambda_t(\tau)$ transform into

$$S_t(\tau)Q^{1/2}(\tau)G'(t_0,\tau)l = \lambda_t(\tau)S_{0t}Q_0^{1/2}l;\ \ S'_0S_0 = I; S'(\tau)S(\tau) \equiv I \tag{30}$$

[2] The formulation of Assumption 3.1 is identical in Parts I and II and so is the assigned numeration "Assumption 3.1".

and

$$\lambda_t(\tau) = (l, G(t_0,\tau)Q(\tau)G'(t_0,\tau)l)^{1/2}/(l, Q_0 l)^{1/2}. \tag{31}$$

Here the known functions used for calculating $S_t(\tau), \lambda_t(\tau)$ *do not depend on* t. Therefore, the unknown functions $S_t(\tau), \lambda_t(\tau)$ *do not depend on* t either, no matter what is the interval $[t_0, t]$. Therefore, the lower indices t in S_{0t}, S_t, λ_t may be dropped.

Differentiating (27) in view of the last remark, we come to

$$\dot{Q}_- = A(t)Q_- + Q_- A'(t) + \dot{Q}_*' Q_* + Q_*' \dot{Q}_*, \tag{32}$$

where

$$Q_*(t) = S_0 Q_0^{1/2} G'(t, t_0) + \int_{t_0}^{t} S(\tau) Q^{1/2}(\tau) G'(t,\tau) d\tau,$$

$$\dot{Q}_*(t) = S(t)Q^{1/2}(t), Q_*(t_0) = S_0 Q_0$$

Using the notation

$$H(t) = Q_*^{-1}(t)S(t)Q^{1/2}(t) = Q_*^{-1}(t)\dot{Q}_*(t), \tag{33}$$

we further come to equation

$$\dot{Q}_- = A(t)Q_- + Q_- A'(t) + H'(t)Q_-(t) + Q_-(t)H(t), \;\; Q(t_0) = Q_0. \tag{34}$$

The differentiation of (21) also gives

$$\dot{q_-} = A(t)q_- + q(t), \;\; q(t_0) = q_0. \tag{35}$$

This leads to the following theorem.

Theorem 3.1 *Under Assumption 3.1 the solution to Problem 2.1 is given by ellipsoid* $\mathcal{E}(q_-(t), Q_-(t))$ *where* $Q_-(t), q_-(t)$ *are given by equations (34),(35), and the functions* $S(t), \lambda(t)$ *involved in the calculation of* $H(t)$ *satisfy together with* S_0 *the relations (30),(31), where the lower indices* t *in* S_{0t}, S_t, λ_t *are to be dropped.*

Lemma 3.1 *Function* $H(t) = Q_*^{-1}(t)S(t)Q^{1/2}(t)$ *in (33) may be also expressed through equation*

$$\dot{Q}_* = Q_* A'(t) + S(t)Q^{1/2}(t), \; Q_*(t_0) = S_0 Q_0^{1/2}. \tag{36}$$

This gives the next result.

Lemma 3.2 *The ellipsoid* $\mathcal{E}(q_-(t), Q_-(t))$ *of Theorem 3.1 given by equations (35), (34), (33), (36), depends on the selection of the orthogonal matrix function* $S(t)$ *and for any such* $S(t)$ *the inclusion*

$$\mathcal{E}(q_-(t), Q_-(t)) \subseteq \mathcal{X}[t], \;\; t \geq t_0,$$

is true with equality (26) attained under conditions (30),(31).

Let us now suppose that $l(t)$ of Problem 2.1 is the vector function that generates *any* continuous curve of related *support vectors* on the surface of $\mathcal{X}[t]$. Then one has to use formula (27), having in mind that $S_{0t}, S_t(\tau)$ depend on t. After a differentiation of (27) in t, one may observe that (34) transforms into

$$\dot{Q}_- = A(t)Q_- + Q_-A'(t) + H'(t)Q_-(t) + Q_-(t)H(t) + \Psi(t,\cdot), \quad Q(t_0) = Q_0. \tag{37}$$

where

$$\Psi(t,\cdot) = G(t,t_0)\left(Q_0^{1/2}(\partial S'_{0t}(t_0)/\partial t) + \int_{t_0}^t G(t_0,\tau)Q^{1/2}(\tau)(\partial S'_t(\tau)/\partial t)d\tau\right)' \times$$
$$\left((\partial S_{0t}(t_0)/\partial t)Q_0^{1/2} + \int_{t_0}^t \partial(S_t(\tau)/\partial t)Q^{1/2}(\tau)G'(t_0,\tau)d\tau\right)G'(t,t_0).$$

Lemma 3.3 *Under Assumption 3.1 the functional* $\Psi(t,\cdot) \equiv 0$.

Similarly to Section 3 we come to the proposition.

Theorem 3.2 *Let $l(t)$ generate a curve $x^*(t)$ of related support vectors for the sets $\mathcal{X}[t]$ that form a system trajectory of (19) due to some control $u(t)$. Then Assumption 3.1 is satisified ($l(t)$ is a "good" curve) and the functional* $\Psi(t,\cdot) \equiv 0$.

We will now demonstrate the internal approximations for the system of Example I. [3]

4 Example II

Consider again the system of Section 5, Part I (see formula (53) of Part I).

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u,$$

$$x_1(0) = x_1^0, \; x_2^0(0) = x_2^0; \;\; |u| \le \mu, \quad \mu > 0.$$

Here

$$x_1(t) = x_1^0 + x_2^0 t + \int_0^t (t-\tau)u(\tau)d\tau,$$

$$x_2(t) = x_2^0 + \int_0^t u(\tau)d\tau.$$

Assume $X^0 = \mathcal{B}_\epsilon(0) = \{x : (x,x) \le \epsilon^2\}$. Then the support function

$$\rho(l|\mathcal{X}[t]) = \max\{(l, x(t))|\; |u| \le \mu, \; x^0 \in X^0\}$$

[3] Example I was given in Part I of this publication.

of the reach set $\mathcal{X}[t] = \mathcal{X}(t, 0, X^0)$ may be calculated directly and is given by

$$\rho(l|\mathcal{X}[t]) = \epsilon(l_1^2 + (l_1 t + l_2)^2)^{1/2} + \int_0^t |l_1(t-\tau) + l_2| d\tau,$$

The boundary $\partial\mathcal{X}[t]$ of the reach set $\mathcal{X}[t]$ may be calculated from formula (54) of Part I.

Solving the problem for any $t > 0$, let us set $l^0 = l(t)$.Then

$$x_1(t) = \epsilon l_1(t)/(l_1^2(t) + (l_1(t)t + l_2(t))^2)^{1/2} \ \pm\ \mu(t^2/2 - (tl_1(t) - l_2(t))^2/l_1^2(t)), \tag{38}$$

$$x_2(t) = \epsilon(l_1(t)t + l_2(t))/(l_1(t)^2 + (l_1(t)t + l_2(t))^2)^{1/2} \pm 2\mu(l_2 - tl_1(t))/l_1(t) \pm \mu t,$$

Proceeding further, we shall select $l(t)$ satisfying Assumption 3.1, namely, as $l(t) = e^{-A't}l^*$. This transforms here into $l_1(t) = l_1^*,\ l_2(t) = l_2^* - tl_1^*$ and (38) transforms into

$$x_1(t) = \epsilon l_1(t)/(l_1^{*2} + l_2^{*2})^{1/2} \ \pm\ \mu(t^2/2 - (tl_1^* - l_2^{*2}(t))^2/l_1^{*2}(t)), \tag{39}$$

$$x_2(t) = \epsilon l_2/(l_1^{*2} + l_2^{*2})^{1/2} \pm 2\mu(l_2^* - tl_1^*)/l_1^* \pm \mu t.$$

The last relations depend only on the two-dimensional vector l^*. They produce a parametric family of curves $\{x_1(t), x_2(t)\}$ that cover all the surface of the reach tube $\mathcal{X}[t]$ so that vectors $x(t) = \{x_1(t), x_2(t)\}$ are the points of support for the hyperplanes generated by vectors $l(t) = \{l_1^*, -tl_1^* + l_2^*\}$. The reach tube that starts at $X^0 \neq 0$ with these curves on its surface is shown in fig.1.

Let us now construct the tight internal ellipsoidal approximations for $\mathcal{X}[t]$ that touch the boundary $\partial\mathcal{X}[t]$ *from inside* at points of support taken for a given vector $l = l^*$.

The support function $\rho(\mathcal{X}[t])$ may be rewritten as

$$\rho(\mathcal{X}[t]) = \epsilon(l^*, Q(t)l^*)^{1/2} + \mu \int_0^t (l^*, Q(\tau)l^*)^{1/2} d\tau, \tag{40}$$

where

$$Q(\tau) = \begin{pmatrix} \tau^2, & -\tau \\ -\tau, & 1 \end{pmatrix}$$

and $Q^{1/2}(\tau) = Q(\tau)(1 + \tau^2)^{-1/2}$.

According to (27) and in view of Assumption 3.1, we have (taking $S_0 = I$),

$$Q_-(\tau) = \left(\epsilon I + \int_0^t Q^{1/2}(\tau)S'(\tau)d\tau\right)' \left(\epsilon I + \int_0^t S(\tau)Q^{1/2}(\tau)d\tau\right), \tag{41}$$

where matrix $S(\tau)$ must satisfy the conditions

$$S'(\tau)S(\tau) = I, \ \ S(\tau)Q^{1/2}(\tau)l^* = \epsilon\lambda(\tau)l^*, \ \ \tau \geq 0. \tag{42}$$

for some $\lambda(\tau)$ and calculations give

$$\epsilon\lambda^2(\tau) = (l^*, Q(\tau)l^*)(l^*, l^*)^{-1} = (l_1^*\tau - l_2^*)^2(l_1^{*2} + l_2^{*2})^{-1}. \tag{43}$$

Denote

$$p(\tau) = Q^{1/2}(\tau)l^* = \begin{pmatrix} \tau^2 l_1^* - \tau l_2^* \\ -\tau l_1^* + l_2^* \end{pmatrix} (1+\tau^2)^{-1/2} = r_p(1+\tau^2)^{-1/2} \begin{pmatrix} \cos\phi_p(\tau) \\ \sin\phi_p(\tau) \end{pmatrix},$$

where

$$r_p(\tau) = |l_1^*\tau - l_2^*|(1+\tau^2)^{1/2}, \ \ \phi_p(\tau) = \pm\arccos(\tau^2 l_1^* - \tau l_2^*)/r_p = \arccos(\tau/(1+\tau^2)^{1/2})$$

and also

$$l^* = (l^*, l^*)^{1/2} \begin{pmatrix} \cos\phi_l(\tau) \\ \sin\phi_l(\tau) \end{pmatrix}, \ \ \phi_l = \pm\arccos(l_1^*/(l_1^{*2} + l_2^{*2})^{1/2}).$$

Selecting further the orthogonal matrix-valued function $S(\tau)$ as

$$S(\tau) = \begin{pmatrix} \cos\alpha(\tau), -\sin\alpha(\tau) \\ \sin\alpha(\tau), \cos\alpha(\tau) \end{pmatrix},$$

we may rewrite the second relation of (42) as

$$\begin{pmatrix} \cos(\phi_p(\tau) + \alpha(\tau)) \\ \sin(\phi_p(\tau) + \alpha(\tau)) \end{pmatrix} r_p(\tau)(1+\tau^2)^{1/2} = \epsilon\lambda(\tau)(l_1^{*2} + l_2^{*2})^{1/2} \begin{pmatrix} \cos\phi_l(\tau) \\ \sin\phi_l(\tau) \end{pmatrix}. \tag{44}$$

where $\tau \in [t_0, t]$. Here $\alpha(\tau)$ has to be selected from the equality

$$\phi_p(\tau) + \alpha(\tau) = \phi_l(\tau), \ \ \tau \in [t_0, t], \tag{45}$$

and $\lambda(\tau)$ is given by (43).

Equations (44), (45) need no recalculation for new values of t.

Thus we have found an orthogonal matrix function $S(\tau)$

$$S(\tau) = \begin{pmatrix} \cos(\phi_l(\tau) - \phi_p(\tau)), -\sin(\phi_l(\tau) - \phi_p(\tau)) \\ \sin(\phi_l(\tau) - \phi_p(\tau)), \cos(\phi_l(\tau) - \phi_p(\tau)) \end{pmatrix},$$

that depends on l^*, is continuous in τ and satisfies (44).

Matrix $Q_-(t)$ may now be calculated from the equations

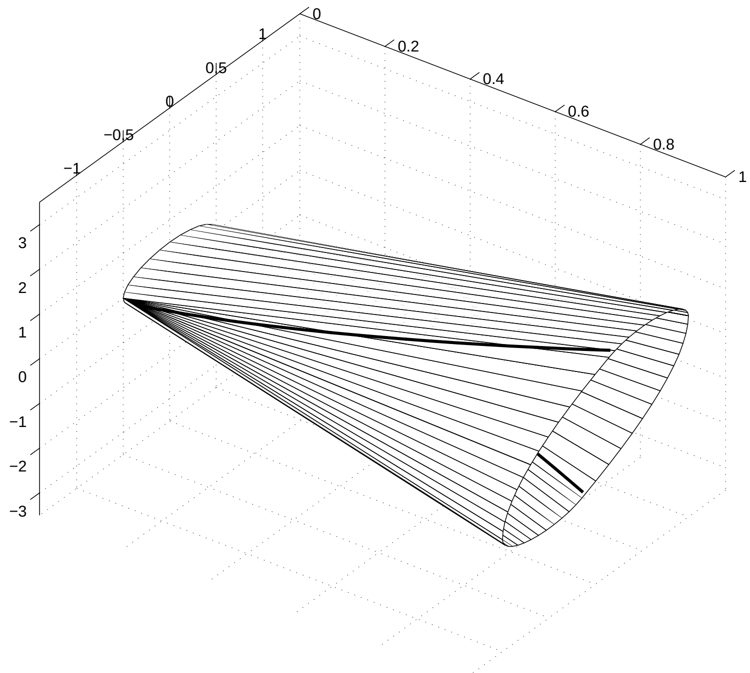
$$\dot{Q}_- = \dot{Q}_*'(t)Q_*(t) + Q_*(t)\dot{Q}_*(t), \ \ Q(0) = \epsilon^2 I,$$

where

$$\dot{Q}_* = S'(t)Q^{1/2}(t), \ \ Q_*(0) = \epsilon I.$$

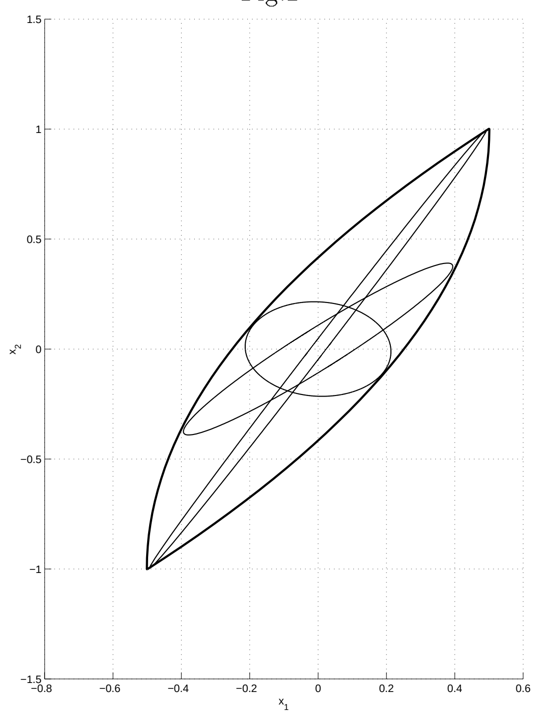
The internal ellipsoids for the reach set $\mathcal{X}[t] = \mathcal{X}(t, 0, X^0)$ are shown in figures 2-4 for $X^0 = \mathcal{E}(0, \epsilon I)$ with epsilon increasing from $\epsilon = 0$ (fig.2) to $\epsilon = 0.175$ (fig.3), and $\epsilon = 1$ (fig.4). The tube in fig.1 corresponds to the epsilon of fig.3. One may also observe the exact reach sets, $\mathcal{X}(t, 0, 0)$ taken for $\epsilon = 0$, inside the sets $\mathcal{X}(t, 0, X^0) = \mathcal{X}[t]$ in figures 3,4.

Fig.1



0
0.2
0.4
0.6
0.8
1
1
0.5
0
−0.5
−1
3
2
1
0
−1
−2
−3

Fig.2



1.5
1
0.5
0
−0.5
−1
−1.5
x_2
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
x_1

Fig.3

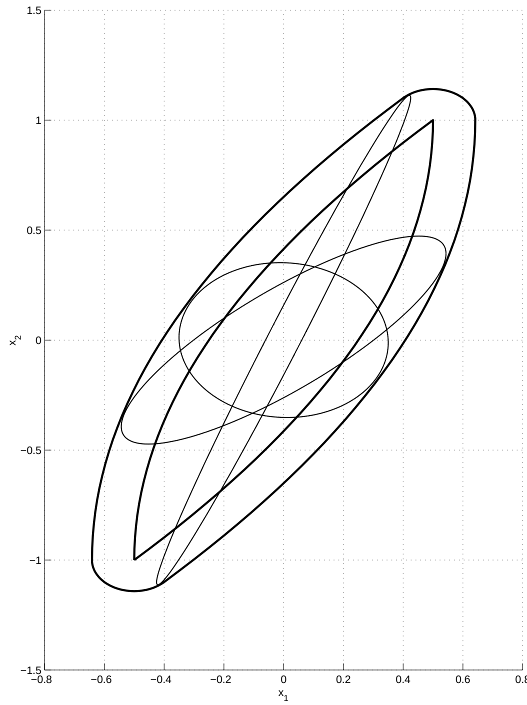
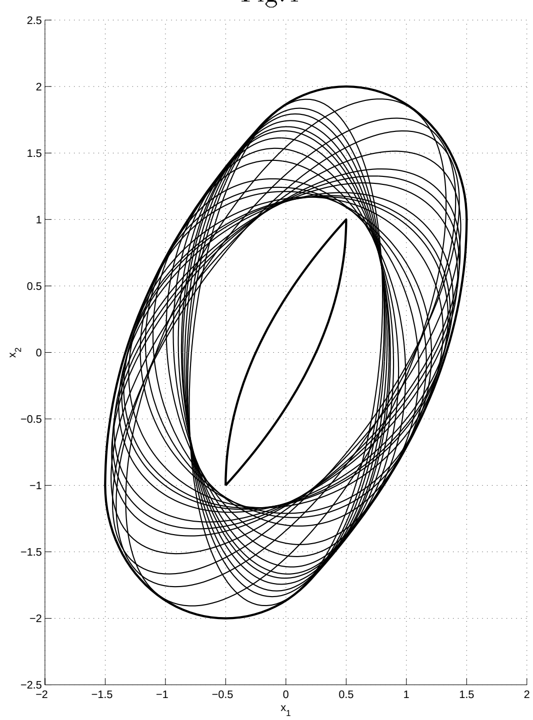

1.5
1
0.5
0
−0.5
−1
−1.5
x2
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
x1

Fig.4


2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
x2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x1

We will now pass to the description of ellipsoidal approximations for systems with box-valued constraints on the controls and initial values.

5 Box-valued constraints

Let us now assume that system (19) is subjected to hard bounds of the "box" type, namely,

$$u(t) \in \mathcal{P}(t), \quad x(t_0) \in \mathcal{X}^0,$$

where

$$\begin{gathered} \mathcal{P}(t) = \{u \in \mathbb{R}^m : |u_i - u_i^0| \leq \mu_i(t)\}, \quad \mu_i(t) \geq 0, \\ \mathcal{X}^0 = \{x \in \mathbb{R}^n : |x_j - x_j^0| \leq \nu_j\}, \quad \nu_j \geq 0 \\ i = \{1, \dots, m\}, j = \{1, \dots, n\}. \end{gathered} \tag{46}$$

and u_i^0, x_j^0 are given.

Will it be possible to use ellipsoidal approximations for the respective reach sets now, that $\mathcal{P}, \mathcal{X}^0$ are not ellipsoids? To demonstrate that this is indeed possible, we proceed as follows.

Let us define a box $\mathcal{P}$ with center p as $\mathcal{P} = \mathcal{B}(p, P)$ where $P = \{p^{(1)}, \dots, p^{(n)}\}$ is an array of n vectors ("directions") $p^{(i)}$ such that

$$\mathcal{B}(p, P) = \{x : x = p + \sum_{i=1}^{n} p^{(i)} \alpha_i, \quad \alpha_i \in [-1, 1]\}.$$

Then box $\mathcal{P}(t)$ of (46) may be presented as $\mathcal{P}(t) = \mathcal{B}(u^0(t), P(t))$, where $P = \{p^{(1)}, \dots, p^{(n)}\}$, $p^{(i)} = \mu_i(t)\mathbf{e}^{(\mathbf{i})}$ and $\mathbf{e}^{(\mathbf{i})}$ is a unit ort oriented along the axis $0x_i$. Box $\mathcal{B}(u^0(t), P(t))$ is a rectangular parallelepiped.

A linear transformation T of box $\mathcal{B}(p, P)$ will give

$$T\mathcal{B}(p, P) = \mathcal{B}(Tp, TP)$$

Thus, in general, box $\mathcal{B}(Tu^0(t), TP(t))$ will not be rectangular. Let us now approximate a box by a family of ellipsoids.

Taking set $\mathcal{B}(0, P)$, we may present it as the sum of m degenerate ellipsoids $\mathcal{E}(0, Q_{ii})$, where

$$Q_{ii} = q_{ii}\mathbf{e}^{(\mathbf{i})}\mathbf{e}^{(\mathbf{i})\prime}, \quad q_{ii} = \mu_i^2$$

Here Q_{ii} is a diagonal matrix with diagonal elements $q_{kk} = 0, k \neq i, \quad q_{ii} = \mu_i^2$, (its only nonzero element is $q_{ii} = \mu_i^2$).

Then

$$\mathcal{B}(0, P) = \sum_{i=1}^{m} \mathcal{E}(0, Q_{ii}) \subseteq \mathcal{E}(0, Q(p)),$$

where $p = \{p_1, \ldots, p_m\}$ and

$$Q(p) = \Big(\sum_{i=1}^{m} p_i\Big)\Big(\sum_{i=1}^{m} p_i^{-1} Q_{ii}\Big), \tag{47}$$

These relations were usually used for nondegenerate ellipsoids,(see [13], section 2.7). However, the application of Lemma 3.2.1 , [13], indicates that it is also true for the degenerate case. (The proof is similar to the one in [13]).

Given vector $l \in \mathbb{R}^m$, take $p = \{p_1, \ldots, p_m\}$ as

$$p_i = |l_i|\mu_i \;\; if \quad l_i \neq 0, \tag{48}$$

$$p_i = \epsilon^2 (m-k)^{-1} (\sum_{i=1}^{m} |l_i|\mu_i)^{-1/2} ||l||^2 \;\; if \quad l_i = 0.$$

Here $||l||^2 = \sum_{i=1}^{n} l_i^2$, k is the number of nonzero coordinates $l_j \neq 0$ of l. Then, selecting p as in the previous lines, we have, assuming $p_i \neq 0$ for $i = 1, ..., k$ and $p_i = 0$ for $i > k$,

$$\rho^2(l|Q(p)) = (l, Q(p)l) = (\sum_{i=1}^{m} p_i)(l, (\sum_{i=1}^{m} Q_{ii} p_i^{-1} l) \tag{49}$$

$$= (\sum_{i=1}^{k} |l_i|\mu_i + \sum_{i=k+1}^{m} p_i)(\sum_{i=1}^{k} |l_i|\mu_i) = (\sum_{i=1}^{k} |l_i|\mu_i)^2 + \epsilon^2 ||l||^2$$

$$\geq (\sum_{i=1}^{k} |l_i|\mu_i)^2 = \rho^2(l|\mathcal{B}(0, Q)),$$

so that

$$\rho^2(l|Q(p)) - \rho^2(l|\mathcal{B}(0, Q)) \leq \epsilon^2 ||l||^2. \tag{50}$$

Note that with $\epsilon > 0$ the matrix $Q(p)$ is nondegenerate.

However, if we allow $\epsilon = 0$ and take p as in (48), then (50) will turn into an equality, but $Q(p)$ will be *degenerate*. The set $\mathcal{E}(0, Q(p))$ will be an elliptical cylinder.

Theorem 5.1 *(i) An external ellipsoidal approximation*

$$\mathcal{B}(0, P) \subseteq \mathcal{E}(0, Q(p))$$

is given by ellipsoid $\mathcal{E}(0, Q(p))$, where $Q(p)$ is given by (47).

(ii)With p selected according to (48), the inequality (50) will be true. If one takes $\epsilon = 0$ in (48), then (50) turns into an equality. However, the ellipsoid $\mathcal{E}(0, Q(p))$ then becomes degenerate (an elliptical cylinder).

A similar approximation is true for box $\mathcal{B}(0, X) = \mathcal{X}^0$.

Lemma 5.1 *Under a linear transformation T we have*

$$T\mathcal{B}(0, \mathbf{Q}) \subseteq \mathcal{E}(0, TQ(p)T') \tag{51}$$

This follows directly from the above.

6 Integrals of box-valued functions

Consider a set - valued integral

$$\int_{t_0}^{\tau} \mathcal{B}(0, B(t)P(t))dt \tag{52}$$

and a partition Σ_N similar to the one of Section 2, Part I. Here $P(t)$ is an $m \times m$ diagonal matrix, as before, $B(t)$ is a continuous $n \times m$ matrix.

Then

$$\int_{t_0}^{\tau} \mathcal{B}(0, B(t)P(t))dt = \lim \sum_{i=1}^{N} \sum_{j=1}^{m} \mathcal{E}(0, B(t_i)Q_{jj}(t_i))B'(t_i))\sigma_i$$

with $N \to \infty, \sigma_N \to 0$. Applying again the formula for the external ellipsoidal aprroximation of the sum of ellipsoids, we have

$$\sum_{i=1}^{N} \sum_{j=1}^{m} \mathcal{E}(0, B(t_i)Q_{jj}(t_i))B'(t_i))\sigma_i \subseteq \mathcal{E}(0, X_{+N}(p_N[\cdot])),$$

$$X_{+N}(p_N(\cdot))) = \left(\sum_{i=1}^{N} \sum_{j=1}^{m} p_j(t_i) \right) \left(\sum_{i=1}^{N} \sum_{j=1}^{m} p_j^{-1}(t_i) Q_{jj}(t_i) \right), \;\; p_j(t_i) > 0.$$

Here $p_N[\cdot] = \{p_j(t_i) \mid j = 1, \ldots, m, \;\; i = 1, \ldots, N\}$.

Taking $p_j(t_i)$ to be the values of continuous functions $p_j(t), \quad j = 1, \ldots, m$, and passing in the previous relation to the limit with $N \to \infty, \quad \sigma[N] \to 0$, we come to the next conclusion.

Lemma 6.1 *The following inclusion is true*

$$\int_{t_0}^{\tau} \mathcal{B}(0, B(t)P(t))dt \subseteq \mathcal{E}(0, X_+(\tau, p[\cdot])dt \tag{53}$$

$$X_+(p[\cdot]) = \sum_{j=1}^{n} \left(\int_{t_0}^{\tau} p_j(t)dt \right) \left(\sum_{j=1}^{n} \int_{t_0}^{\tau} p_j^{-1}(t) B(t) Q_{jj}(t) B'(t) dt \right)$$

for any continuous functions $p_j(t) > 0$.

Here $p[\cdot] = \{p_j(\cdot) \mid j = 1, \ldots, m, \; t \in [t_0, \tau]\}$.

For a nonrectangular box $T(t)\mathcal{B}(0, P(t)) = \mathcal{B}(0, T(t)P(t))$ and a nonzero box $T_0\mathcal{B}(0, X^0) = \mathcal{B}(0, T_0X^0)$ in a similar way we have

Theorem 6.1 *The following inclusion is true*

$$\mathcal{X}[\tau] = \mathcal{B}(0, T_0X^0) + \int_{t_0}^{\tau} \mathcal{B}(0, T(t)B(t)P(t))dt \subseteq \mathcal{E}(0, X_+(\tau, p[\cdot])) \tag{54}$$

where

$$X_+(\tau, p[\cdot]) = \left(\sum_{k=1}^{n} p_k^{(0)} + \sum_{j=1}^{m} \int_{t_0}^{\tau} p_j(t)dt\right) \times \tag{55}$$

$$\times\left(\sum_{j=1}^{n} \nu_k p_k^{(0)-1} T_0 X_{kk}^0 T_0' + \sum_{j=1}^{m} \int_{t_0}^{\tau} \mu_j p_j^{-1}(t) T(t) B(t) Q_{jj}(t) B'(t) T'(t) dt\right) = X_+(\tau, p[\cdot]).$$

Here $p[\cdot] = \{p_k^{(0)},\ p_j(\cdot) \mid\ k = 1, \ldots, n;\ \ j = 1, \ldots, m,\ t \in [t_0, \tau]\}$.

In order that an equality

$$\rho(l|\mathcal{B}(0, T_0 X^0)) + \int_{t_0}^{\tau} \rho(l|\mathcal{B}(0, T(t)B(t)P(t)))dt = \rho(l|\mathcal{E}(0, X_+(\tau, p[\cdot]))) \tag{56}$$

would be possible for a given $l \in \mathbb{R}^n$, we would formally have to choose $X_+(\tau, p[\cdot])$ taking

$$p_j(t) = \mu_j(l, T(t)B(t)Q_{jj}(t)B'(t)T'(t)l)^{1/2}, \quad p_k^{(0)} = \nu_k(l, T_0 X_{kk}^0 T_0' l)^{1/2}. \tag{57}$$

But a nondegenerate matrix $X_+(\tau, p[\cdot])$ would be possible only if $p_j(t) \neq 0$ almost everywhere and $p_k^{(0)} \neq 0$. The equality is then checked by direct calculation.

Lemma 6.2 *In order that for a given* $l \in \mathbb{R}^n$ *there would be an equality (56), it is necessary and sufficient that* $p_j(t),\ \ p_k^{(0)}$ *would be selected according to (57) and both of the conditions* $p_j(t) \neq 0$ *almost everywhere and* $p_k^{(0)} \neq 0$ *would be true.*

Otherwise, either an equality (56) will still be ensured, but with a degenerate $\mathcal{E}(0, X_+(\tau, p[\cdot]))$, or, for any ϵ given in advance, an inequality

$$\rho^2(l|X_+(\tau, p[\cdot])) - (\rho(l|\mathcal{B}(0, T_0 X^0)) + \int_{t_0}^{\tau} \rho(l|\mathcal{B}(0, T(t)B(t)P(t)))dt)^2 \leq \epsilon^2 ||l||^2 \tag{58}$$

may be ensured with a nondegenerate $\mathcal{E}(X_+(\tau, p[\cdot]))$.

This may be done by selecting

$$p[\cdot] = p^{\epsilon}[\cdot] = \{p_k^{(0\epsilon)},\ p_j^{(\epsilon)}(\cdot) \mid\ k = 1, \ldots, n;\ \ j = 1, \ldots, m,\ t \in [t_0, \tau,]\}$$

as

$$p_j^{(\epsilon)}(t) = \mu_j(l, T(t)B(t)Q_{jj}(t)B'(t)T'(t)l)^{1/2} + (\epsilon^2 ||l||^2)\left(m(\tau - t)\sum_{j=1}^{m} \mu_j(l, T(t)B(t)Q_{jj}(t)B'(t)T'(t)l)^{1/2}\right)^{-1},$$

$$p_{kk}^{(0\epsilon)} = \nu_k(l, T_0 X_{kk}^0 T_0' l)^{1/2} + (\epsilon^2 ||l||^2)(n \sum_{k=1}^{n} \nu_k(l, T_0 X_{kk}^0 T_0' l)^{1/2})^{-1}.$$

It may be useful to know when $p_j(t) \neq 0$ almost everywhere .

Lemma 6.3 *In order that $p_j(t) = (l, T(t)B(t)Q_{jj}(t)B'(t)T'(t)l)^{1/2} \neq 0$ almost everywhere, for all $l \in \mathbb{R}^n$, it is necessary and sufficient that functions $T(t)B(t)\mathbf{e}^{(j)}$ would be linearly independent . (The j-th column of $T(t)B(t)$ would consist of linearly independent functions).*

This follows from the definition of linearly independent functions.

Note that with $\epsilon = 0$ we have

$$\left(\sum_{k=1}^{n} p_k^{(0)} + \sum_{j=1}^{m} \int_{t_0}^{\tau} p_j(t)dt\right) = (l, X_+(\tau, p[\cdot])l)^{1/2}, \tag{59}$$

The parameters of the ellipsoid $\mathcal{E}(0, X_+(\tau, p[\cdot]))$ may be expressed through a differential equation.

Taking $X_+[\tau] = X_+(\tau, p[\cdot])$, differentiate it in τ. We get

$$\dot{X}_+ = \left(\sum_{j=1}^{m} p_j(\tau)\right)\left(\sum_{k=1}^{n} p_k^{-1} T_0 X_{kk} T_0' + \sum_{j=1}^{m} \int_{t_0}^{\tau} p_j^{-1}(t)T(t)B(t)Q_{jj}(t)B'(t)T'(t)dt\right)$$

$$+\left(\sum_{k=1}^{n} p_k^{(0)-1} + \sum_{j=1}^{m} \int_{t_0}^{\tau} p_j(t)dt\right)\left(\sum_{j=1}^{m} p_j^{-1}(\tau)T(\tau)B(\tau)Q_{jj}(\tau)B'(\tau)T'(\tau)\right)$$

Denoting

$$\pi_j(\tau) = p_j(\tau)\left(\sum_{k=1}^{n} p_k^{(0)} + \sum_{j=1}^{m} \int_{t_0}^{\tau} p_j(t)dt\right)^{-1} = \mu_j |l_j| (l, X_+[\tau]l)^{-1/2},$$

$$X_+^0 = \sum_{k=1}^{n} \pi_k^{(0)-1} T_0 X_{kk} T_0, \quad \pi_k^{(0)} = p_k^{(0)}\left(\sum_{k=1}^{n} p_k^{(0)}\right)^{-1},$$

rearranging the coefficients similarly to Section 3 of Part I, we come to

$$\dot{X}_+[\tau] = \left(\sum_{j=1}^{m} \pi_j(\tau)\right) X_+[\tau] + \sum_{j=1}^{m} \pi_j^{-1}(\tau)T(\tau)B(\tau)Q_{jj}(\tau)B'(\tau)T'(\tau), \quad X_+[t_0] = X_+^0 \tag{60}$$

Remark 6.1. If boxes $\mathcal{B}(p(t), P(t)), \mathcal{B}(x^0, X^0)$ have nonzero centers $p(t)$, then all the previous relations hold with centers of ellipsoids changing from 0 to $x^0(t)$, where

$$\dot{x}^0 = B(t)p(t), \quad x(t_0) = x^0,$$

so that $\mathcal{E}(0, X_+(\tau, p[\cdot]))$ turns into $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$.

Theorem 6.2 *(i) The matrix* $X_+(\tau, p[\cdot])$ *of the external ellipsoid* $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$. *that ensures the inclusion (54) satisfies the differential equation and the tinitial condition (60).*

(ii) The ellipsoid $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$ *ensures the equality (56) (namely,* $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$ *touches set* $\mathcal{X}[\tau]$ *of (54) along the direction l), if parameters* $p[\cdot]$ *are selected as in (57) and* $\pi_j(t)$ *are defined respectively, for all* $t \in [t_0, \tau]$, *with* $\pi_k^{(0)} \neq 0$. *Then* $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$ *is nondegenerate for any l.*

(iii) In order that $\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))$ *would be nondegenerate for all l, (with box* $\mathcal{B}(x^0, X^0) = \{x^0\}$ *being a singleton), it suffices that functions* $T(t)B(t)\mathbf{e}^{(j)}$ *would be linearly independent on* $[t_0, t_1]$.

(iv) In general, for any given ϵ, *selecting* $\pi_j^{(\epsilon)}(t)$, $\pi_k^{(0\epsilon)}$ *similarly to* $\pi_j(t), \pi_k^{(0)}$, *but with* $p_j(t)$, $p_k^{(0)}$ *substituted for* $p_j^{(\epsilon)}(t)$, $p_k^{(0)\epsilon)}$, *one is able to ensure the inequality (58).*

We may now proceed with the approximation of reach sets for system (19).

7 Reach tubes for box-valued constraints. External approximations

Consider system (19) under box-valued constraints (46). Its reach set will be

$$\mathcal{X}^*[t] = G(t, t_0)\mathcal{X}[t]$$

where

$$\mathcal{X}[t] = \mathcal{B}(x^0, X^0) + \int_{t_0}^{t} G(t_0, s)B(s)\mathcal{B}(u^0(s), P(s))ds, \tag{61}$$

Let us first apply the results of the previous section to the approximation of $\mathcal{X}[t]$. Taking $T(s) = G(t_0, s), T_0 = I$, we have

$$\mathcal{X}[t] \subseteq \mathcal{E}(x^0(t), X_+[t])),$$

where

$$\dot{X}_+[t] = \left(\sum_{j=1}^{m} \pi_j(t)\right) X_+[t] + \sum_{j=1}^{m} \pi_j^{-1}(t) G(t_0, t)B(t)Q_{jj}(t)B'(t)G'(t_0, t), \tag{62}$$

with initial condition

$$X_+[t_0] = \sum_{k=1}^{n} \pi_k^{(0)} X_{kk}^0, \tag{63}$$

and with $x^0(t)$ evolving due to equation

$$\dot{x}^0 = A(t)x^0 + B(t)u^0(t), \quad x(t_0) = x^0. \tag{64}$$

Further on, denoting $X_+^*[t] = G(t, t_0)X_+[t]G'(t, t_0)$, we obtain

$$\mathcal{X}_+^*[t] \subseteq \mathcal{E}(x^0(t), G(t, t_0)X_+[t]G'(t, t_0)) = \mathcal{E}(x^0(t), X_+^*[t]), \tag{65}$$

where now

$$\dot{X}_+^*[t] = A(t)X_+^* + X_+^*A'(t) + \left(\sum_{j=1}^{m} \pi_j^*(t)\right) X_+[t] + \sum_{j=1}^{m} \pi_j^{*-1}(t)B(t)Q_{jj}(t)B'(t),; \; X_+^0 = X_+^*[t_0]. \tag{66}$$

Theorem 7.1 *The inclusion (65) is true, whatever be the parameters* $\pi_j(t) > 0, \quad \pi_k > 0$ *of the equation (66).*

Let us now presume that Assumption 3.1 is fulfilled :the vector function $l(t)$ along which we would like to ensure the tightness property is taken as $l(t) = G(t_0, t)l, \ \ l \in \mathbb{R}^n$.

Then, following the schemes of sections 3, 4 , Part I, we come to the following results.

Theorem 7.2 *Under Assumption 3.1, in order that the equality*

$$\rho(l|\mathcal{X}[\tau]) = \rho(l|\mathcal{E}(x^0(\tau), X_+(\tau, p[\cdot]))$$

would be true for a given "direction" l*, the external ellipsoids* $\mathcal{E}(x^0(\tau), X_+(\tau, p[\cdot]))$ *should be taken with*

$$\pi_j^*(t) = \frac{(l, G(t_0,t)B(t)Q_{jj}(t)B'(t)G'(t_0,t)l)^{1/2}}{(l, X_+^*[t]l)^{1/2}}, \ \ t_0 \le t \le \tau, \tag{67}$$

$$X_+^{*0} = \left(\sum_{k=1}^{n} \nu_k |l_k|\right)\left(\sum_{k=1}^{n} (\nu_k|l_k|)^{-1} X_{kk}^0\right),$$

provided $\pi_j^*(t) > 0$ *almost everywhere and* $|l_k| > 0, \ \ \forall k = \{1, \ldots, n\}$.

Otherwise, taking for any given $\epsilon > 0$ *the parameters*

$$\pi_j^{*\epsilon}(t) = \frac{\mu_j(l, G(t_0,t)B(t)Q_{jj}(t)B'(t)G'(t_0,t)l)^{1/2} + \epsilon^2||l||^2\gamma^{-1}(t)}{(l, X_+^*[t]l)^{1/2} + \epsilon||l||}, \tag{68}$$

$$\gamma(t) = \sum_{j=1}^{m} \mu_j(l, G(t_0,t)B(t)Q_{jj}(t)B'(t)G'(t_0,t)l)^{1/2},$$

and

$$X_+^*[t_0] = X_+^{*0\epsilon} = \sum_{k=1}^{n} \pi_k^{(0\epsilon)-1} X_{kk}^0,$$

where

$$\pi_k^{(0\epsilon)} = p_k^{(0\epsilon)}\left(\sum_{k=1}^{n} (p_k^{(0\epsilon)-1}\right),$$

$$p_k^{0\epsilon} = \nu_k(l, X_k^0 l)^{1/2} + \epsilon^2||l||^2\left(n\sum_{k=1}^{n} \nu_k(l, X_k^0 l)^{1/2}\right)^{-1}.$$

should be taken instead of $\pi_j^*(t), \pi_k^{(0\epsilon)}$.

An inequality

$$\rho^2(l\ |\mathcal{E}(x^0(t), X_+(\tau, p[\cdot]))) - \rho^2(l|\mathcal{X}[\tau]) \le \epsilon^2||l||^2$$

will then be true.

Remark 7.1. The exact reach set $\mathcal{X}^*[t]$ is the sum of two sets:

$$\mathcal{X}^*[t] = \mathcal{X}_0^*[t] + \mathcal{X}_u^*[t],$$

where $\mathcal{X}_0^*[t] = G(t,t_0)\mathcal{B}(x^0, X^0)$ is a (nonrectangular) box and

$$\mathcal{X}_u^*[t] = \int_{t_0}^{t} G(t,s)B(s)\mathcal{B}(u^0, P(s))ds,$$

is a convex compact set. Set $\mathcal{X}_0^*[t]$ cannot be exactly approximated by nondegenerate ellipsoids (see figures 5,6 in the next Section), while set $\mathcal{X}_u^*[t]$ may be represented exactly by nondegenerate ellipsoids under the condition (iii) of Theorem 6.2. Let us reformulate this condition.

Theorem 7.3 *In order that for any $l \in \mathbb{R}^n$ an equality*

$$\rho(l|\mathcal{X}_u^*[t]) = \rho(l|\mathcal{E}(0, X_+^*[t]))$$

would be possible for an appropriately selected ellipsoid $\mathcal{E}(0, X_+^*[t])$, it is sufficient that the pair $\{A(t), \mathbf{e}^k\}$ would be completely controllable for any $k = 1, ..., m$.

Then $X_+^*[t]$ will be correctly defined when described by equation (66), with parameters $\pi_j^*(t)$ and initial condition $X_+^*[t_0] = X_+^{*0}$ selected due to (67).

This follows from the definition of complete controllability , [15]. Under this condition the boundary of set $X_+^*[\tau]$ will not have any "platforms" and it can be totally decribed by "tight" ellipsoids, as in Part I (Sections 1-4, figures 1-4).

Finally, a parametric presentation of set $\mathcal{X}^*[t]$, similar to (51), (52), can be produced. Then

$$x^*(t) = x^0(t) + X_+^*[t]l^*(l^*, X_+^*[t]l^*)^{1/2}, , \tag{69}$$

with

$$X^*(t_0) = x^0 + X_+^0 l(l, X_+^0 l)^{1/2}, \;\; l^*(t) = G(t,t_0))'l. \tag{70}$$

with either $\pi_j(t)$, π_k^0 or $\pi_j^\epsilon(t)$, $\pi_k^{(0\epsilon)}$ selected as indicated in Theorem 11.2. This results in an array of external ellipsoids of either type $\mathcal{E}(x^0(t), X_+^*[\tau])$ or type $\mathcal{E}(x^0(t), X_+^{*\epsilon}[t])$, which lead to equalities of type (56), or inequalities of type (58) accordingly.

8 Example III

.

In this section we consider external ellipsoids for reach sets and tubes that are initiated from given starting sets rather than starting points, including box-valued starting sets. In order to compare reach sets initiated by ellipsoids with those generated by boxes, consider system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega^2 x_1 + u_2,$$

with $|u(t)| \leq \mu$ and either (a) $x(0) \in \mathcal{X}_0 = \mathcal{E}(x_0, X^0)$, or (b) $x(0) \in \mathcal{X}_0 = \mathcal{B}(x_0, P^0)$.

Here $\mathcal{X}_0$ is the starting set which is either an ellipsoid $\mathcal{E}(x_0, X^0)$ or a box $\mathcal{B}(x_0, P^0)$, x_0 is a given vector, matrix $X^0 = X'^0 > 0$, and matrix P^0 is with positive coefficients.(Further we take $X^0 = I, P^0 = I$, to be a unit matrix).

We have:

$$G(t,\tau) = G(t-\tau) = \begin{pmatrix} \cos\omega(t-\tau), \omega^{-1}\sin\omega(t-\tau)) \\ -\omega\sin(\omega(t-\tau)), \cos\omega(t-\tau)) \end{pmatrix},$$

$$x_1(t) = \cos\omega t\ x_1^0 + \omega^{-1}\sin\omega t\ x_2^0 + \int_0^t \omega^{-1}\sin\omega(t-\tau)d\tau,$$

$$x_2(t) = -\omega\sin\omega t\ x_1^0 + \cos\omega t\ x_2^0 + \int_0^t \cos\omega(t-\tau)d\tau,$$

which yields the support function (b' = {0,1})

$$\rho(l|\mathcal{X}[t]) = \rho(l|G(t,0)\mathcal{X}_0 + \int_0^t |l'G(t,\tau)b|d\tau.$$

The tube $\mathcal{X}[t],\ \ t \geq 0$, may now be approximated by ellipsoids. Assumption 3.1 of Parts I,II requires that the "good" curves along which we calculate the reach sets are of the form $l(t) = G'(-t)l$. Then, for case (a), we may use the results of Part I (equations (25), (26) of Part I, where $Q(t)$ is substituted by $Q(t) = \mu^{1/2}b'b = \mu^{1/2}$, see Remark 1.1 of Part I). With $\omega^2 = 1, \mu = 1, x_0 = 0, X^0 = I$, the calculations are illustrated in Fig.5 (for the reach sets at instants $t = 0.5, t = 1$) and Fig.6 (for the reach tube).

For case (b) the calculations are made due to relations of (64,66), and of Theorem 7.1, with $m = 1, n = 2$ and are illustrated in Figures 7, 8 for the reach sets and the reach tube accordingly.

The next two figures 9, 10 are again related to case (b) but with $\mathcal{B}(u^0, P(t)) \equiv \{0\}$, so that the reach sets $\mathcal{X}*[t] = \mathcal{X}_0^*[t]$ are box - valued for all t.

Finally, in figures 11, 12, we return to Example I of Part I and construct the reach sets and reach tube that are initiated from box $\mathcal{X}^0 = \mathcal{X}[t_0] = \{x : |x_i| \leq 1,\ \ i = 1, 2\}$.

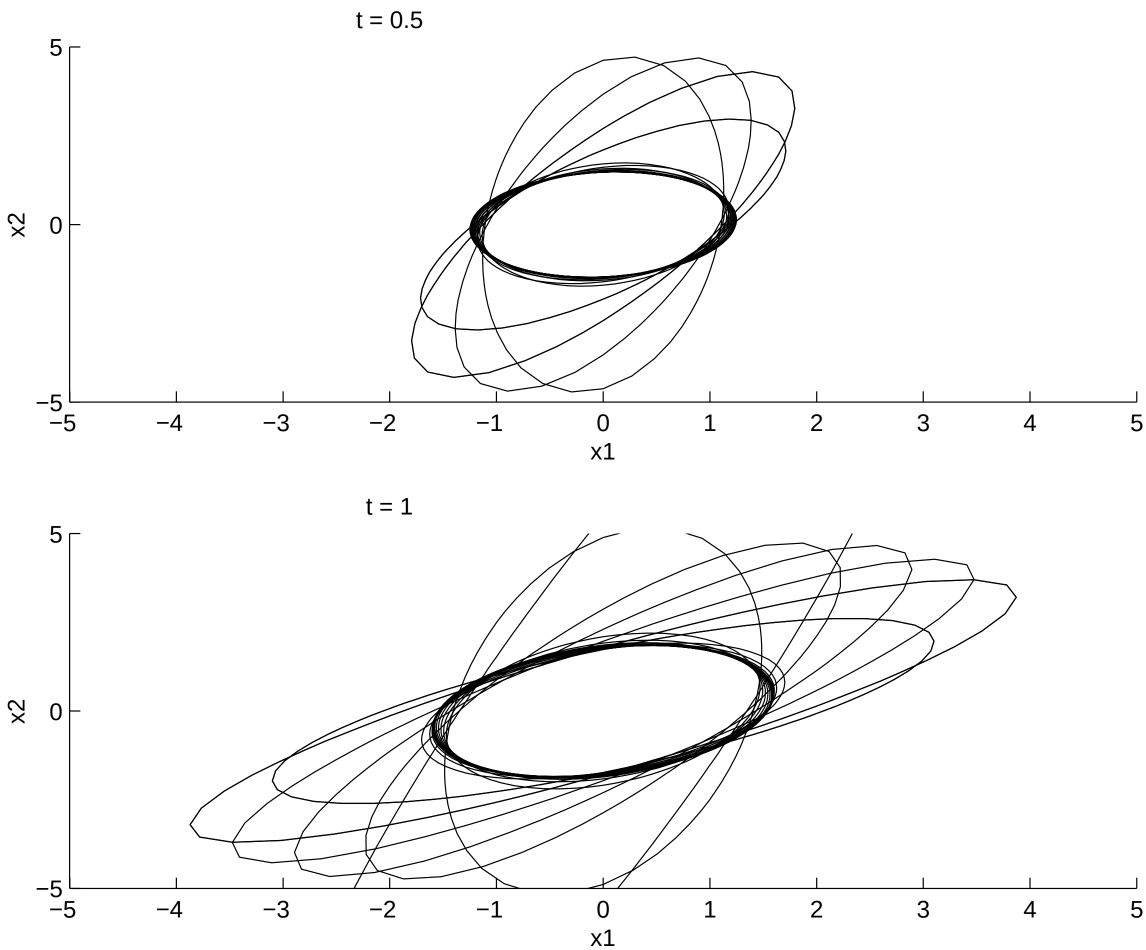


Fig.5

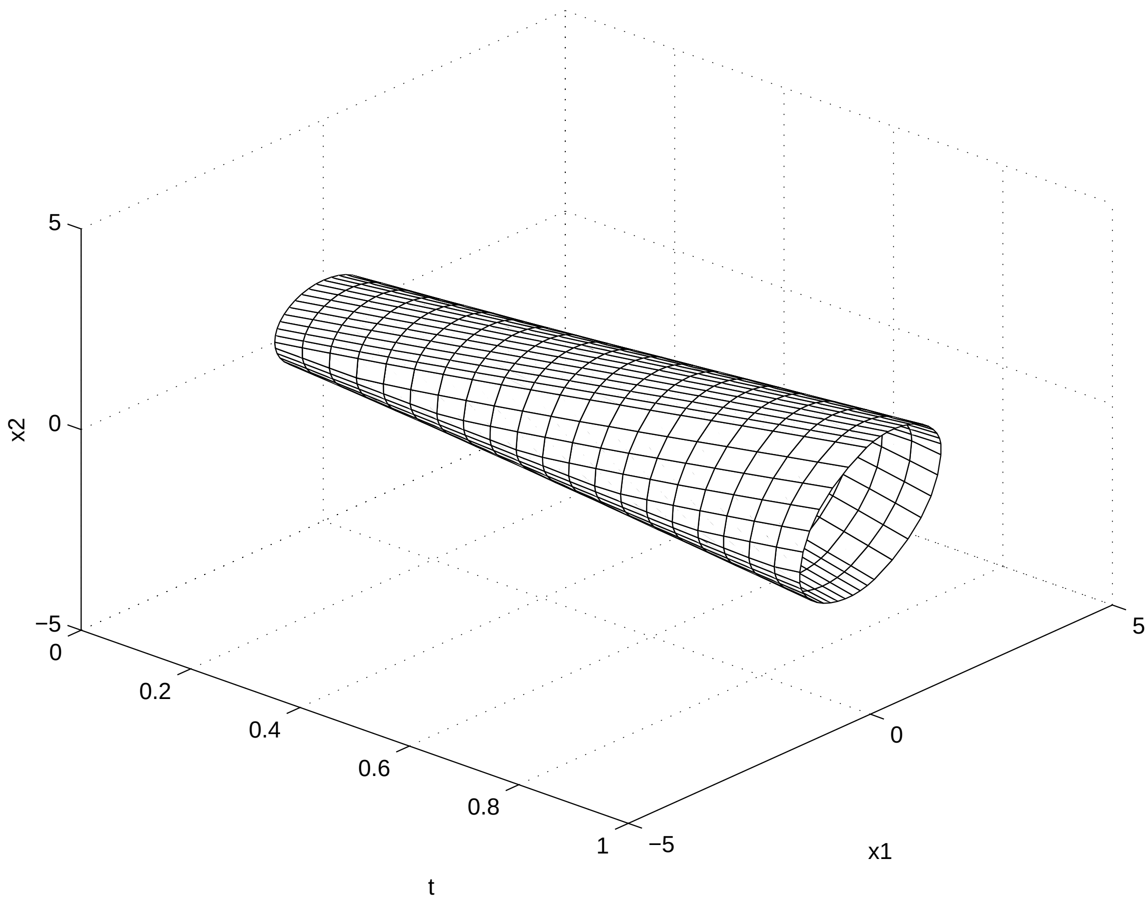


Fig.6

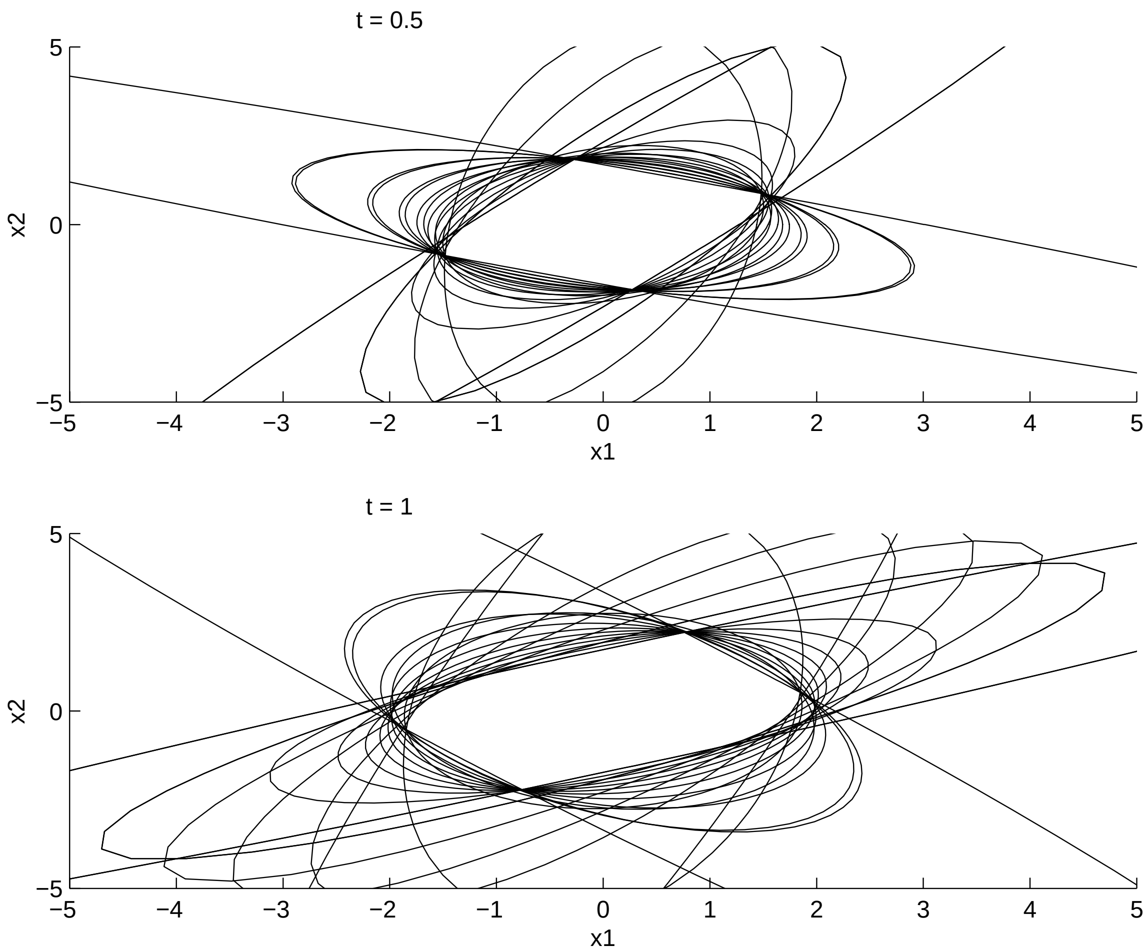


Fig.7

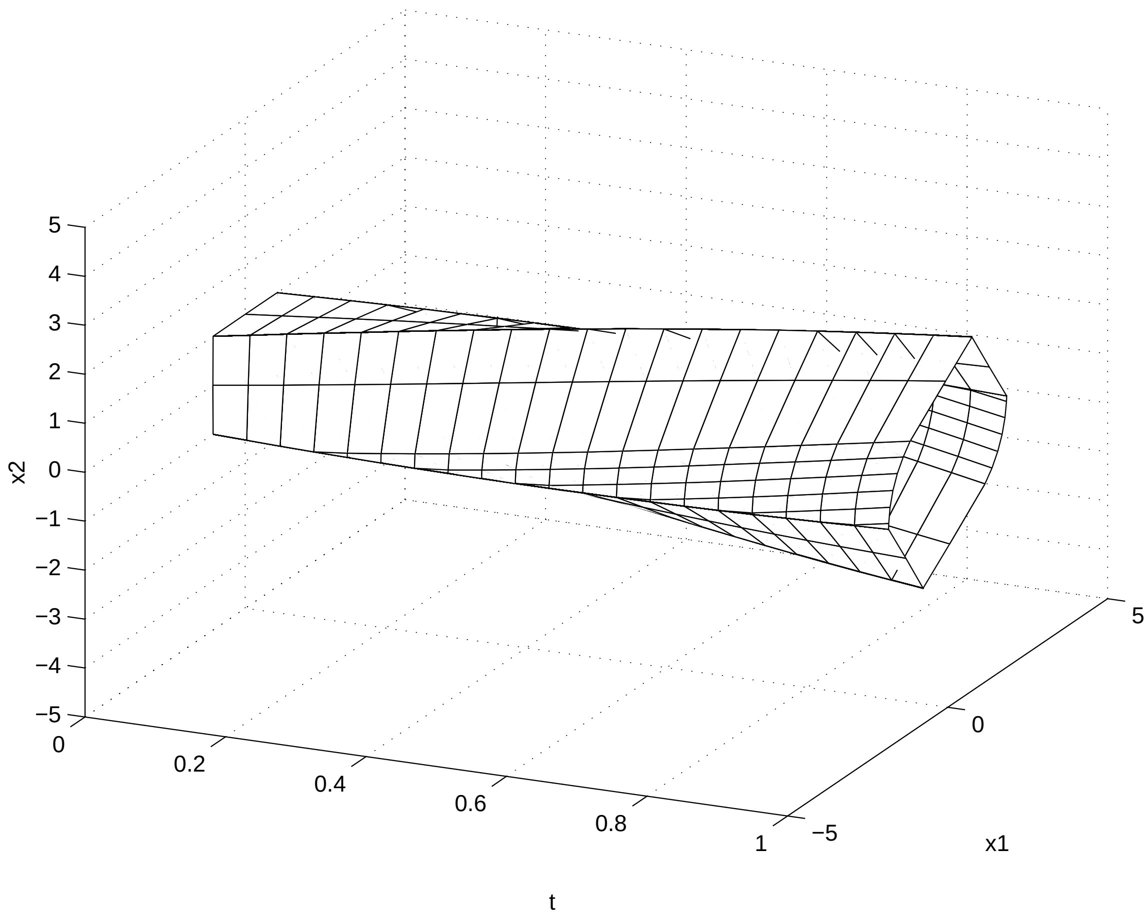


Fig.8

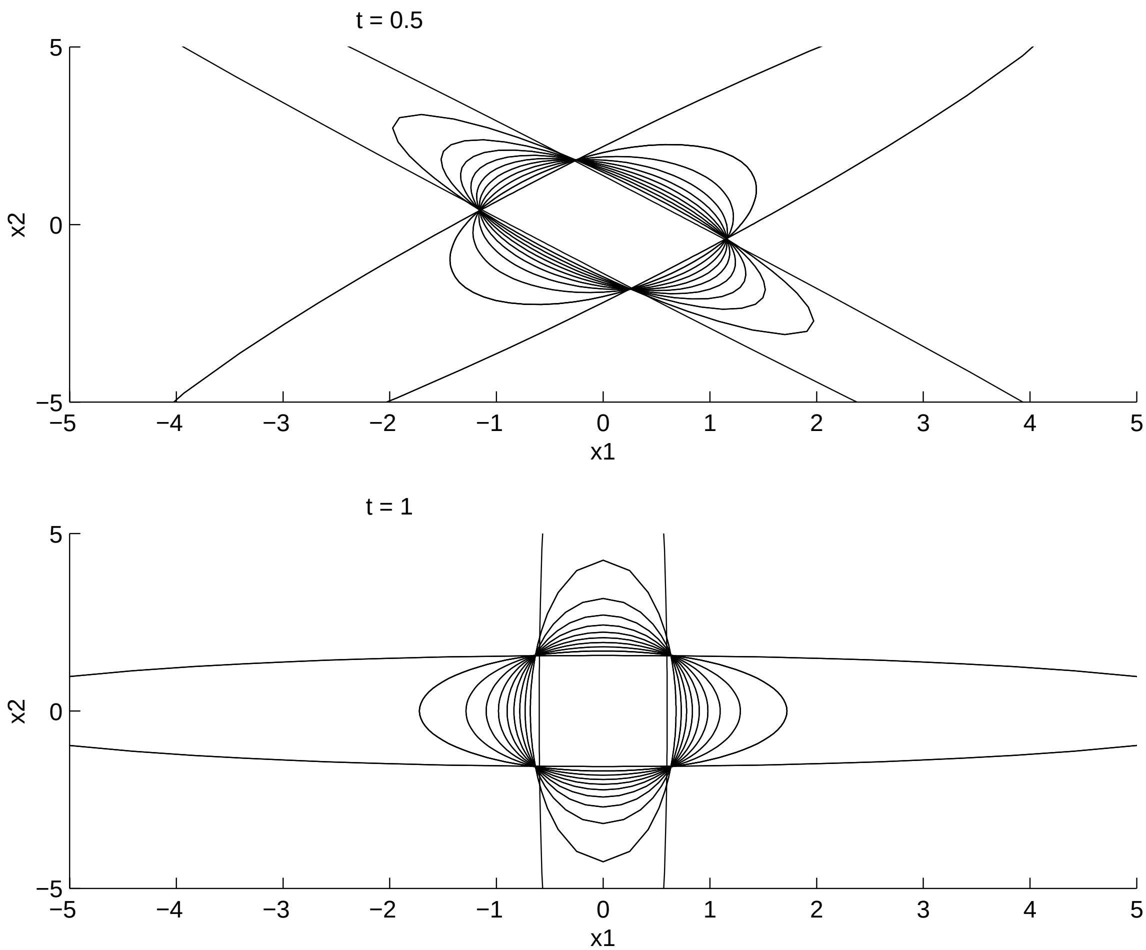


Fig.9

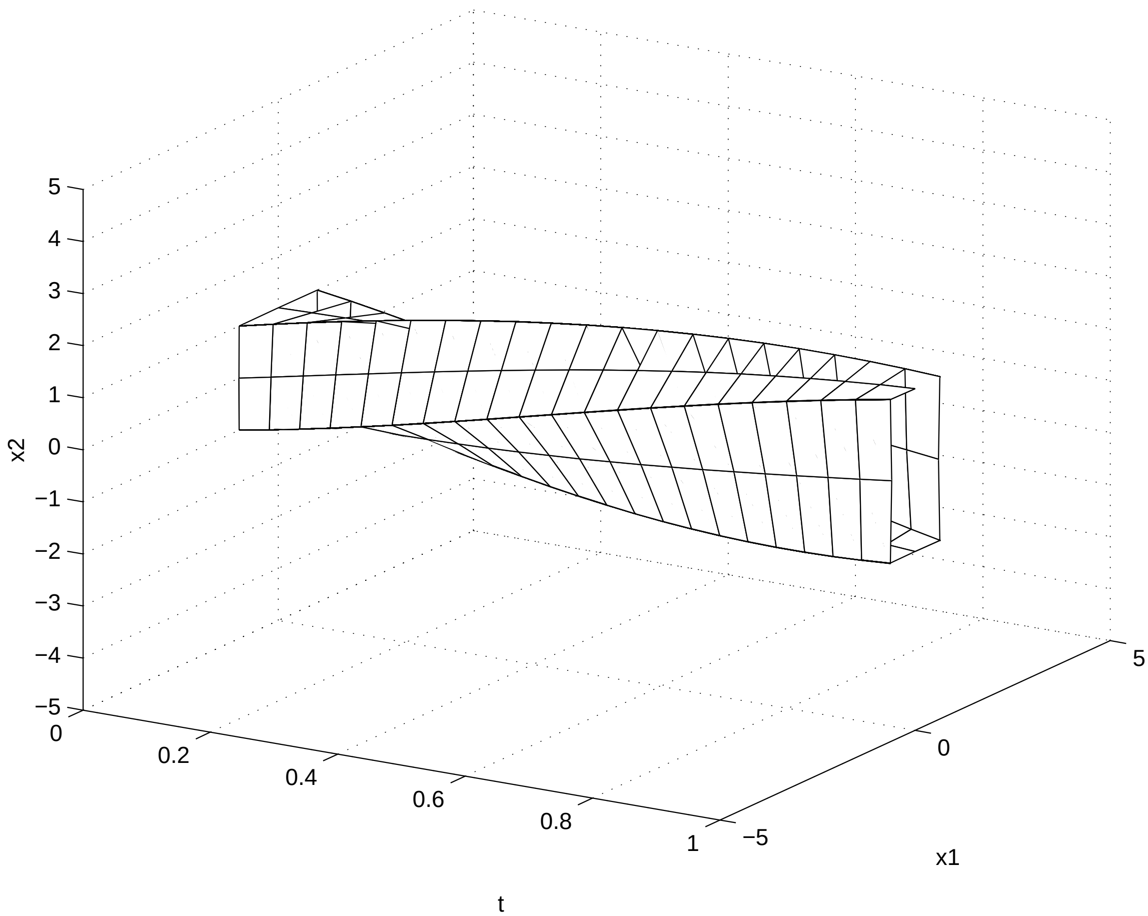


Fig.10

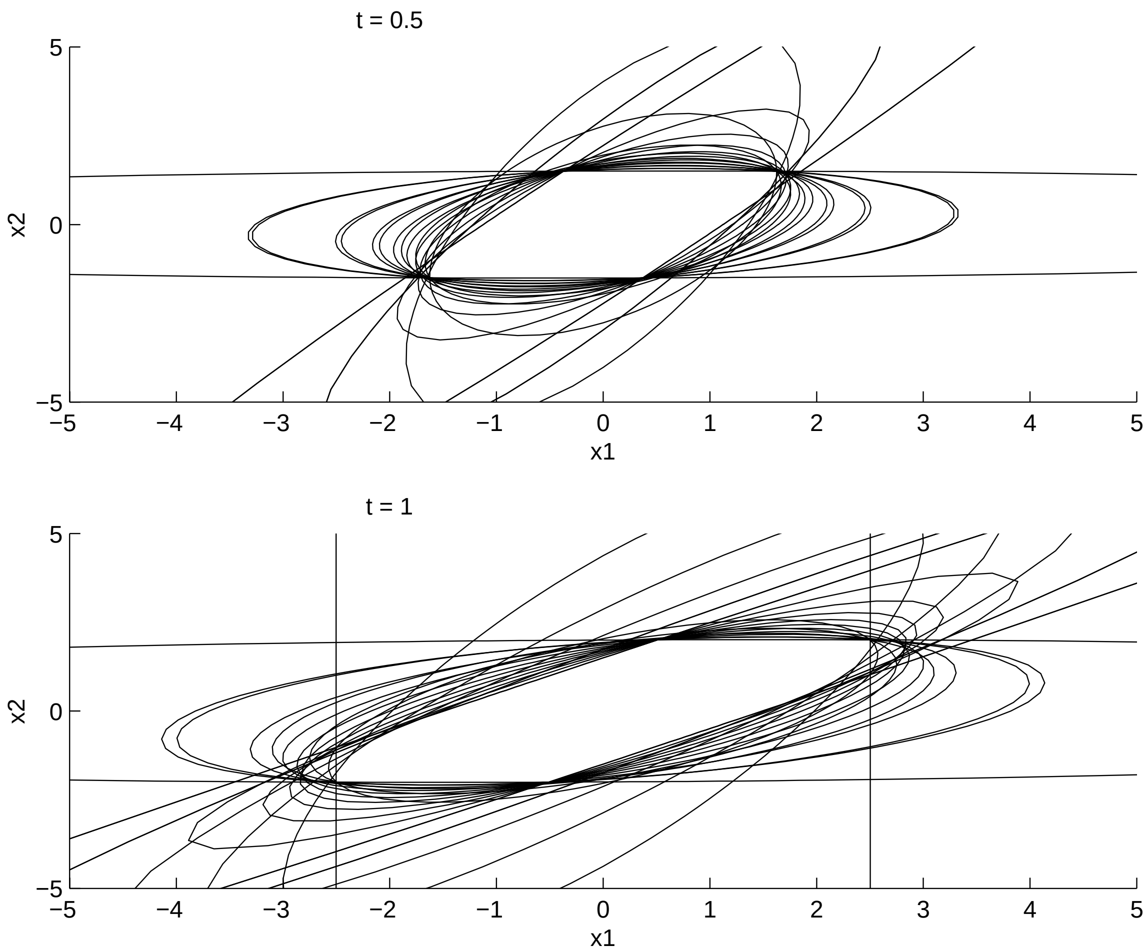


Fig.11

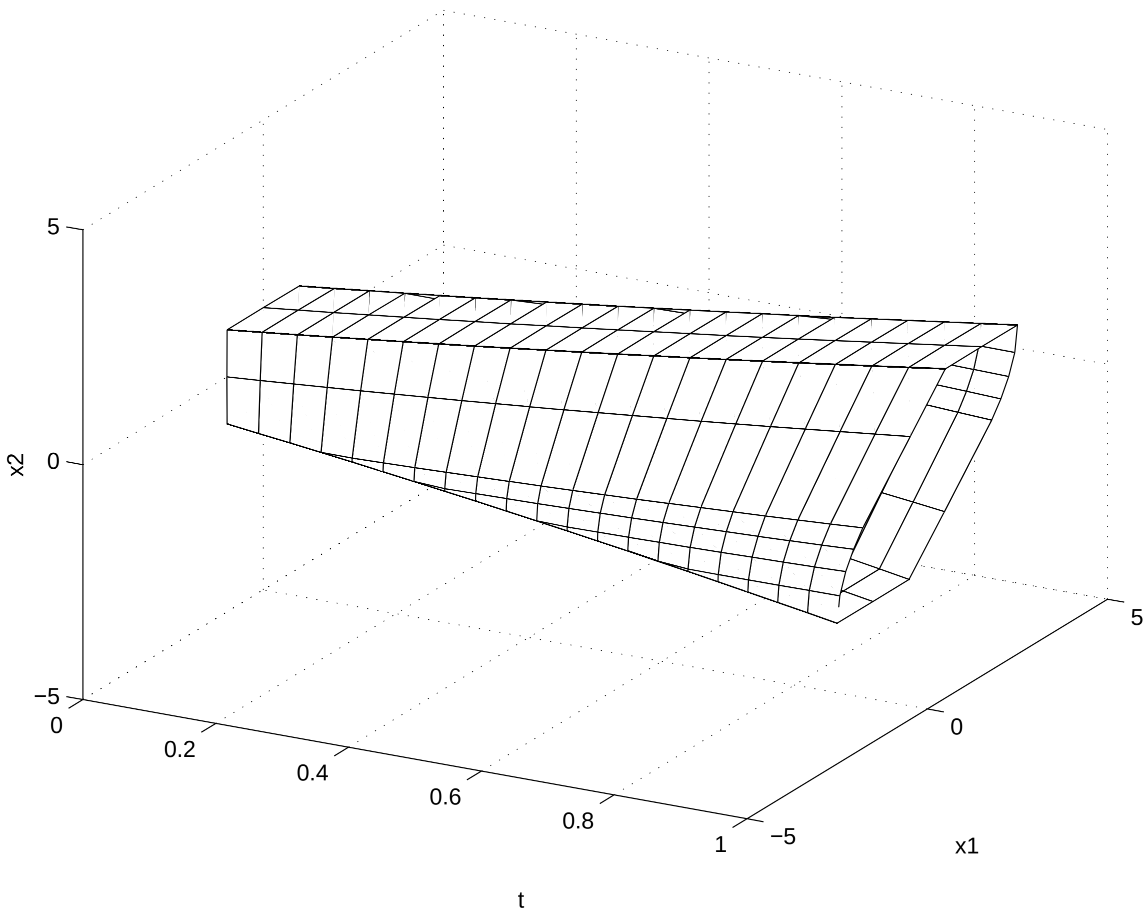


Fig.12

9 Reach tubes for box-valued constraints. Internal approximations

Following Remark 2.3, we recall that the results of Sections 1-3 are all true for degenerate ellipsoids. We may therefore directly apply them to box-valued constraints

$$\mathcal{P}(t) = \mathcal{B}(u^0(t), P(t)), \;\; \mathcal{X}^0 = \mathcal{B}(x^0, X^0),$$

using relations (9), (27), in view of the inclusions

$$\mathcal{B}(u^0(t), P(t)) = \sum_{j=1}^{m} \mathcal{E}(u^0(t), Q_{ii}(t)), \; \mathcal{B}(x^0, X^0) = \sum_{k=1}^{n} \mathcal{E}(x^0, X_{kk}),$$

where

$$Q_{ii}(t) = q_{ii}\mathbf{e}^{(\mathbf{i})}\mathbf{e}^{(\mathbf{i})\prime}, \;\; q_{ii} = \mu_i^2, \;\; X_{kk}^0 = x_{kk}^0\mathbf{e}^{(\mathbf{k})}\mathbf{e}^{(\mathbf{k})\prime}, \;\; x_{kk}^0 = \nu_k^2,$$

and $\mathbf{e}^{(\mathbf{i})}, \mathbf{e}^{(\mathbf{k})}$ are unit orts in the respective spaces $\mathbb{R}^m, \mathbb{R}^n$.This leads to the following statement.

Theorem 9.1 *(i) An ellipsoid* $\mathcal{E}(x^0(t), X^*_-(t))$ *that satisfies the relations*

$$X^*_-(t) = G(t,t_0)X_-(t)G'(t,t_0),$$

where

$$X_-(t) = \left(\sum_{k=1}^{n} (X^0_{kk})^{1/2} S'_{0k} + \int_{t_0}^{t} \sum_{j=1}^{n} G(t_0,s)(B(s)Q_{jj}(s)B'(s))^{1/2} S'_j(s)ds \right) \times \tag{71}$$

$$\times \left(\sum_{k=1}^{n} S_{0k}(X^0_{kk})^{1/2} + \int_{t_0}^{t} \sum_{j=1}^{n} S_j(s)(B(s)Q_{jj}(s)B'(s))^{1/2} G'(t_0,s)ds \right).$$

and S_{0k}, S_j *are any orthogonal matrices of dimensions* $n \times n$, *(*$S_{0k}S'_{0k} = I, ; S_jS'_j = I$*), is an internal ellipsoidal approximate of the reach set* $\mathcal{X}[t]$ *of (61).*

(ii) In order that for a given "direction" l the equality

$$\rho(l \mid \mathcal{X}[t]) = \rho(l \mid \mathcal{E}(x^0(t), X^*_-(t)))$$

would be true, it is necessary an sufficient that there would exist a vector $d \in \mathbb{R}^n$, *such that the equalities*

$$S_{0k}X^{1/2}_{kk} l = \lambda_{0k} d, \quad S_j(B(s)Qjj(s)B'(s))^{1/2}G'(t_0,s)l = \lambda_j(s)d, \tag{72}$$

$$k = 1,\ldots,n, \quad j = 1,\ldots,m; \quad s \in [t_0,t],$$

would be true for some scalars $\lambda_{0k},\ \lambda_j(s)$.

(iii) The function $x^0(t)$ *is the same as for external approximations, and is given by (64).*

We may now express the relations for $X^*(t)$ through differential equations similar to those of Section 3.

Denote

$$Z(t) = \sum_{k=1}^{n} (X^0_{kk})^{1/2} S'_{0k} + \int_{t_0}^{t} \sum_{j=1}^{m} G(t_0,s)(B(s)Q_{jj}B'(s))^{1/2} S'_j(s)ds.$$

Then

$$X^*_-(t) = Y(t)Y'(t), \quad Y(t) = G(t,t_0)Z(t).$$

Differentiating $X^*_-(t)$ and using the previous relations, we come to the proposition.

Theorem 9.2 *The matrix* $X^*_-(t)$ *of the ellipsoid*

$$\mathcal{E}(x^0(t), X^*_-(t)) \subseteq \mathcal{X}[t],$$

satisfies the equation

$$\dot{X}^*_- = A(t)X^*_- + X^*_-A'(t) + Y(t)S'(t)(B(t)Q_{jj}(t)B'(t))^{1/2} + (B(t)Q_{jj}(t)B'(t))^{1/2}S(t)Y'(t) \tag{73}$$

with initial condition

$$X^*_-(t_0) = (\sum_{k=1}^{n}(X^0_{kk})^{1/2}S'_{0k})'(\sum_{k=1}^{n}(X^0_{kk})^{1/2}S'_{0k}), \qquad (74)$$

where

$$\dot{Y} = A(t)Y + \sum_{j=1}^{m}(B(t)Q_{jj}B'(t))^{1/2}S'_j(t),$$

$$Y(t_0) = \sum_{k=1}^{n}(X^0_{kk})^{1/2}S'_{0k}.$$

and S_{0k}, S_j *are orthogonal matrices of dimensions* $n \times n$, ($S_{0k}S'_{0k} = I$; $S_jS'_j = I$).

In order that the equality $\rho(l\ |\mathcal{X}[t]) = \rho l\ |\mathcal{E}(x^0(t), X^*_-(t)))$ would be true, it is necessary and sufficient that relations of type (72) would be satisfied.

Conclusion

This paper studies the behavior of tight internal ellipsoidal approximations of reach sets and reach tubes. It shows that equation (34) with appropriately chosen parameter $S(t)$ (an orthogonal matrix-valued function restricted by an equality) generates a family of internal ellipsoids that touch the reach tube or its neighborhood *from inside* along a special family of "good" curves that cover the whole tube. Such "good" curves are the same as for the external approximations. The suggested techniques allow on-line calculation of the internal ellipsoids without additional computational burden present in other approaches. The calculation of similar ellipsoids along any other given smooth curve on the boundary of the reach tube requires additional burden as compared with the "good" ones. The internal approximations of this paper rely on relations different from those indicated in [1], [2], [13] and are relevant for solving various classes of control and design problems requiring guaranteed performance.

We would finally like to emphasize that the suggested approach appears to be appropriate for *parallel computations*.

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